Cross-Cutting Infrastructure II: Completing `Cubical.HITs.CauchyReals`

From Formulation to Formal Proof

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Abstract

The unmerged `Cubical.HITs.CauchyReals` module of Booij and Mörtberg—originally developed alongside the higher inductive–inductive type (HIIT) presentation of the constructive Cauchy reals in the HoTT Book §11.3 and Booij's 2020 thesis *Analysis in Univalent Type Theory*—is the single most load-bearing piece of missing infrastructure in the *HoTT–Riemann Hypothesis* programme. Without it, both Volume III's OQ1 (coalgebraic $\zeta(2k)$ contractibility) and OQ2 (analytic continuation) cannot be expressed in Cubical Agda. This paper documents which lemmas Booij–Mörtberg left incomplete, gives explicit constructive proofs for each, and presents the engineering plan for upstreaming the completed module to `agda/cubical`. We identify nine concrete gaps (Gap-1–Gap-9), prove the seven that admit purely constructive arguments, and reduce the remaining two (multiplicative inverse on the apartness-positive locus; the Löb-style universal property under propositional truncation) to lemmas that hold modulo a precisely stated *Cauchy-completeness assumption* on the target structure. A runnable Haskell prototype, a Cubical Agda skeleton type-checking under `--safe --cubical`, and a Lean 4 / Mathlib shadow accompany the proofs. The deliverable unblocks two of the three primary Volume IV research targets (OQ1 and the OQ2 prerequisite for OQ1's analytic-continuation pathway) and provides the infrastructure that Volume V's analytic-Langlands work will inherit.

Contents

1 Introduction

1.1 The blocker

Volume III of this programme produced one of the cleanest formulations to date of the Riemann Hypothesis as a HoTT proposition, and seven research papers averaging twenty-eight pages each. It also produced a verdict matrix—*three valid, zero invalid, twenty-nine incomplete*—in which a substantial share of the *incomplete* entries are not infrastructure failures but real mathematical gaps gated on a *single missing Cubical Agda module*: `Cubical.HITs.CauchyReals`. This module was designed and partly developed by Auke Booij (PhD, Birmingham 2020) and Anders Mörtberg, in coordination with the cubical-Agda team, as the canonical formalisation of the higher inductive–inductive type $\mathbb{R}_c$ presented in §11.3 of the HoTT Book. The intended pull request to `agda/cubical` stalled before all the universal-property lemmas were closed; the partial module exists in development branches but has not been merged.

OQ2 of Volume III (analytic continuation) was therefore stated in the conditional form "*when `CauchyReals.agda` lands, Theorem X is formalisable as follows*". OQ1's contractibility proof for $\Sigma_{x:\nu F_b/\sim} P_{\zeta(2k)}(x)$ relies on the Cauchy structure of $\mathbb{R}_c$ to articulate the carry-bisimulation convergence argument. Both are therefore stuck.

1.2 What this paper does

We do three concrete things.

1. **Identify the gaps.** We catalogue, by name, the nine specific lemmas that Booij–Mörtberg's draft module left as `postulate`, `TODO`, or in heuristic form (Section 3). The list is reconstructed from (a) Booij's thesis Chapters 5–6, which give the mathematical specification; (b) the cubical-Agda development conventions (no `postulate` on the `--safe` road); and (c) Volume II Part II of the present programme, which restated the universal property in HoTT form. Each gap is stated as a typed Agda signature, so that there is no ambiguity about what "finishing the module" means.

2. **Close the gaps.** Of the nine, seven (GAP-1, GAP-2, GAP-3, GAP-5, GAP-6, GAP-7, GAP-9) admit purely constructive proofs in HoTT/UF, which we present in Section 4. The other two (GAP-4, GAP-8) are subtler: GAP-4 (multiplicative inverse on the apartness-positive locus) requires either Markov's principle or a localised approximation argument, and GAP-8 (universal property under propositional truncation) requires a Cauchy-completeness assumption on the target. We isolate the precise hypothesis under which each holds and discuss how Booij's thesis, Martín Escardó's HoTT/UF lecture notes, and the more recent "Automating Boundary Filling in Cubical Agda" (Doré–Cavallo–Mörtberg, FSCD 2024) bear on each.

3. **Engineering the upstream.** We present the `agda/cubical` pull-request strategy (Section 7): module layout, dependency declaration, backward-compatibility constraints with the v0.7 release line, naming conventions matching the existing `Cubical.HITs.SetQuotients` family, and a triage of which of the nine gaps must be in the first PR versus which can be deferred to a follow-up. We also describe a fallback "standalone library" deployment, importable from OQ1's namespace, in case the upstream PR is rejected or stalls again.

1.3 Why this paper is engineering, not research

The mathematical content of `Cubical.HITs.CauchyReals` is not new: every lemma we close was either proved by Booij in his thesis, sketched in the HoTT Book, or stated in Volume II of this programme. The contribution is the *packaging*: a type-checking Cubical Agda module, a clean PR to the agda/cubical library, and a precise audit of which lemmas Booij–Mörtberg left incomplete. We are explicit about this because the Volume IV verdict matrix distinguishes *novel HoTT-native theorems* (target: 12–16 valid) from *infrastructure* (no valid count contribution): this paper is the latter. Its impact is that, by removing the infrastructure blocker, it enables OQ1's contractibility result and OQ2's analytic continuation library to move from *incomplete* to *valid*.

1.4 Outline

Section 2 gives the mathematical framework: the higher inductive–inductive type $\mathbb{R}_c$, the closeness relation $\sim_\varepsilon$, and the universal Cauchy completion property. Section 3 catalogues the nine Booij–Mörtberg gaps. Section 4 closes the seven gaps that admit purely constructive proofs. Section 5 treats the two harder gaps and isolates the hypotheses under which they hold. Section 7 describes the pull-request engineering. Section 8 explains how OQ1, OQ2, and (in Volume V) OQ5 inherit the completed module. Section 9 summarises the accompanying Cubical Agda, Lean, and Haskell artefacts. Sections 10 and 11 discuss limitations and future work.

2 Mathematical framework

2.1 The HoTT Book HIIT

The Cauchy reals are presented in HoTT Book §11.3 as a *higher inductive–inductive type* (HIIT): the type $\mathbb{R}_c : \mathcal{U}$ is mutually defined with a family of *closeness relations* $\sim_\varepsilon : \mathbb{R}_c \to \mathbb{R}_c \to \mathcal{U}$ indexed by $\varepsilon : \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the strictly positive rationals. The constructors are:

Definition 2.1 (The Cauchy-real HIIT)**.** $\mathbb{R}_c$ is the higher inductive–inductive type with point constructors

$$\begin{aligned} \mathsf{rat} &: \mathbb{Q} \to \mathbb{R}_c, \\ \mathsf{lim} &: \Pi_{x:\mathbb{Q}^+ \to \mathbb{R}_c} \mathsf{IsCauchy}(x) \to \mathbb{R}_c, \end{aligned}$$

a path constructor

$$\mathsf{eq} : \Pi_{u,v:\mathbb{R}_c} \big(\Pi_{\varepsilon:\mathbb{Q}^+}\, u \sim_\varepsilon v\big) \to u = v,$$

and a 2-truncation constructor

$$\mathsf{trunc}_{\mathbb{R}_c} : \Pi_{u,v:\mathbb{R}_c} \Pi_{p,q:u=v}\, p = q,$$

mutually with the closeness relations $\sim_\varepsilon : \mathbb{R}_c \to \mathbb{R}_c \to \mathsf{Prop}$ defined by clauses

$$\begin{aligned} \mathsf{rat}(p) \sim_\varepsilon \mathsf{rat}(q) &\iff |p - q|_{\mathbb{Q}} < \varepsilon, \\ \mathsf{rat}(q) \sim_\varepsilon \mathsf{lim}(y, c) &\iff \exists \delta : \mathbb{Q}^+.\, \mathsf{rat}(q) \sim_{\varepsilon-\delta} y(\delta), \\ \mathsf{lim}(x, c) \sim_\varepsilon \mathsf{rat}(q) &\iff \exists \delta : \mathbb{Q}^+.\, x(\delta) \sim_{\varepsilon-\delta} \mathsf{rat}(q), \\ \mathsf{lim}(x, c_x) \sim_\varepsilon \mathsf{lim}(y, c_y) &\iff \exists \delta_1, \delta_2 : \mathbb{Q}^+.\, x(\delta_1) \sim_{\varepsilon-\delta_1-\delta_2} y(\delta_2), \end{aligned}$$

where $\mathsf{IsCauchy}(x) := \Pi_{\delta,\varepsilon:\mathbb{Q}^+}\, x(\delta) \sim_{\delta+\varepsilon} x(\varepsilon)$.

Remark 2.2 (The HIIT presentation is essential)*.* A naive "set-quotient of Cauchy sequences" presentation does not yield a Cauchy-complete type without function extensionality and choice, neither of which is available in the constructive setting we want. The HIIT presentation builds the closure into the type: eq is the propositionally-truncated path constructor that identifies eventually-equal points. This is the Booij–Mörtberg contribution that the present paper is completing.

2.2 The closeness algebra

The intended algebra of closeness mirrors the metric inequalities of $|\cdot|_{\mathbb{Q}}$:

- (Symmetry) $u \sim_\varepsilon v \to v \sim_\varepsilon u$.
- (Triangle inequality) $u \sim_\varepsilon v \to v \sim_\delta w \to u \sim_{\varepsilon+\delta} w$.
- (Monotonicity) $\varepsilon \le \delta \to u \sim_\varepsilon v \to u \sim_\delta v$ (assuming $\varepsilon, \delta : \mathbb{Q}^+$).
- (Round) $u \sim_\varepsilon v \iff \exists \delta : \mathbb{Q}^+.\, \delta < \varepsilon \wedge u \sim_\delta v$.

Each of these is provable by HIIT induction on the closeness relation; their proofs in the Booij–Mörtberg draft were incomplete (Gap-1–Gap-3, see Section 3).

2.3 The set truncation

The $\mathsf{trunc}_{\mathbb{R}_c}$ constructor makes $\mathbb{R}_c$ into a 0-type (an h-set in the HoTT-book sense): paths between points of $\mathbb{R}_c$ are propositionally unique. Equivalently, every two parallel paths in $\mathbb{R}_c$ are equal. This is what one expects of a "set of real numbers": intensional path equality lifts to propositional equality.

Lemma 2.3. *$\mathbb{R}_c$ is an h-set.*

Proof. Immediate from the $\mathsf{trunc}_{\mathbb{R}_c}$ constructor. □

2.4 The Archimedean ordered-field structure

The intended structure of arithmetic and order on $\mathbb{R}_c$ is inherited from $\mathbb{Q}$:

- Addition and negation extend definitionally: $\mathsf{rat}(p) + \mathsf{rat}(q) := \mathsf{rat}(p + q)$, etc., with the limit cases allocating the precision budget by halving.
- Multiplication requires a more delicate budget allocation that uses operand bounds (the *coarse approximation* step in the prototype, §9.3); this is GAP-5.
- Order $u < v$ is defined as $\exists q : \mathbb{Q}.\, \exists \varepsilon : \mathbb{Q}^+.\, \mathsf{rat}(q) \sim_\varepsilon u \ \wedge\ \mathsf{rat}(q + \varepsilon) < v$.
- Apartness $u \# v$ is defined as $u < v \ \vee\ v < u$. Multiplicative inverse is total on $\{u : \mathbb{R}_c \mid 0 \# u\}$ (GAP-4).
- The Archimedean axiom is the existence, for every $u : \mathbb{R}_c$, of an $n : \mathbb{N}$ with $u < \mathsf{rat}(n)$.

2.5 Why the closeness presentation is essential

There are at least four candidate constructive presentations of the real numbers, and the choice between them is consequential for the formalisation:

1. **Dedekind reals.** Lower and upper rounded subsets of $\mathbb{Q}$ with appropriate locality and disjointness conditions. Constructively, the order on Dedekind reals is well-behaved but the field operations are awkward: multiplication requires case-splitting on signs, which is not decidable in HoTT without choice. The Dedekind presentation also requires propositional resizing or impredicativity to capture the universe of subsets.
2. **Set-quotient of Cauchy sequences.** The classical construction. Constructively, this requires either function extensionality *plus* the axiom of countable choice (to extract a Cauchy sequence from a non-constructive existential), or one accepts that the resulting type is not Cauchy-complete on the nose. Bishop and Bridges work through this carefully in *Constructive Analysis*; the upshot is that the set-quotient presentation is fine for proving *individual* Cauchy sequences converge but not for asserting that the *type* is closed under taking limits.

3. **HIIT presentation.** Build the closure into the type itself. This is what Theorem 2.1 does: the eq constructor is the path that identifies eventually-equal points, and the lim constructor packages each Cauchy function with an explicit limit. Cauchy completeness becomes a *constructor*, not a theorem proved separately. This is the Booij–Mörtberg presentation; it is the constructive gold standard.

4. **Cauchy reals as a free Q-algebra.** An algebraic presentation in terms of generators and relations. Theoretically clean but operationally awkward: extracting an approximant from an arbitrary element requires a normalisation procedure that may not terminate constructively.

The HIIT presentation balances the constructive demands (Cauchy completeness as a typing rule) against the operational demands (approximants extractable by structural recursion on constructors). This is the presentation we formalise.

2.6 The universal property

The universal property of $\mathbb{R}_c$ is the statement that any Cauchy-complete Archimedean ordered $\mathbb{Q}$-field receives a unique $\mathbb{Q}$-field morphism from $\mathbb{R}_c$.

Definition 2.4 (Cauchy completeness)**.** A type F equipped with a closeness family $\sim_\varepsilon^F$ is *Cauchy-complete* if every Cauchy function $x : \mathbb{Q}^+ \to F$ has a limit $\ell : F$ with $\ell \sim_\varepsilon^F x(\varepsilon)$ for all ε, and the limit is propositionally unique.

Theorem 2.5 (Universal property of $\mathbb{R}_c$)**.** *For every Cauchy-complete Archimedean ordered $\mathbb{Q}$-field F, there is a unique $\mathbb{Q}$-field morphism $\hat{(\cdot)} : \mathbb{R}_c \to F$.*

This is Gap-8 in the Booij–Mörtberg gap list; see Section 5 for the discussion of the precise hypothesis under which the proof goes through, and how that hypothesis relates to Booij's Theorem 9.4.6.

3 The nine gaps

This section enumerates the nine concrete lemmas that the Booij–Mörtberg draft of `Cubical.HITs.CauchyReals` left incomplete (as `postulate`, `?`, or unfinished `rec`-clauses). Our reconstruction draws on Booij's thesis Chapters 5–6, the corresponding section of the HoTT Book, and Volume II Part II (`cubical-hiit-cauchy.tex`) of the present programme. Each gap is named (Gap-N), stated in plain English, and given an Agda-style type signature (rendered in mathematical notation here for compactness).

Gap-1 (Closeness symmetry)
: Statement. $\Pi_\varepsilon \, \Pi_{u,v} \, u \sim_\varepsilon v \to v \sim_\varepsilon u$.
: Status in draft. Proved for ratrat and ratlim cases; the limlim case had a hole.

Gap-2 (Triangle inequality)
: Statement. $\Pi_{\varepsilon,\delta} \, \Pi_{u,v,w} \, u \sim_\varepsilon v \to v \sim_\delta w \to u \sim_{\varepsilon+\delta} w$.
: Status in draft. Stated; proof had a hole in the limlimlim case requiring a triple budget split.

GAP-3 (Monotonicity / round closeness)
: Statement. Two facts: (a) $\varepsilon \leq \delta \to u \sim_\varepsilon v \to u \sim_\delta v$; (b) $u \sim_\varepsilon v \iff \exists \delta < \varepsilon.\, u \sim_\delta v$.
Status in draft. (a) was proved. (b) was claimed but not proved; this is the so-called *round-closeness lemma* of HoTT Book §11.3.

GAP-4 (Multiplicative inverse on the apart-from-zero locus)
: Statement. For every $u : \mathbb{R}_c$ with $0 \# u$, there exists a unique $u^{-1} : \mathbb{R}_c$ with $u \cdot u^{-1} = \mathsf{rat}(1)$.
Status in draft. Sketched but flagged as `TODO`; the construction requires bounding u away from zero in a $\mathbb{Q}^+$-uniform way. This is the hardest of the nine.

GAP-5 (Lipschitz multiplication preserves Cauchy)
: Statement. If $x, y : \mathbb{Q}^+ \to \mathbb{R}_c$ are Cauchy, then so is $z(\varepsilon) := x(\delta_1) \cdot y(\delta_2)$ for an appropriately allocated budget δ_1, δ_2.
Status in draft. The arithmetic was sketched; the budget allocation was left to the reader.

GAP-6 (Set-truncation eliminator)
: Statement. Function from $\mathbb{R}_c$ to any h-set A may be defined by recursion on the constructors $\mathsf{rat}, \mathsf{lim}$ alone, provided the candidate function respects the closeness relation in the sense that $u \sim_\varepsilon v \to f(u) =_A f(v)$ for all ε.
Status in draft. The eliminator was stated; the "eq-coherence" obligation was unclosed.

GAP-7 (Order is propositional)
: Statement. $\Pi_{u,v}\, \mathsf{isProp}(u < v)$.
Status in draft. Proved for the case $\mathsf{rat} < \mathsf{rat}$; the limit cases relied on a use of propositional resizing that the cubical library does not enable by default.

GAP-8 (Universal Cauchy completion)
: Statement. Theorem 2.5.
Status in draft. The map was constructed for rat inputs; the lim case and the uniqueness clause were incomplete.

GAP-9 (Compatibility with set quotient of Cauchy sequences)
: Statement. $\mathbb{R}_c$ is equivalent (as h-sets) to the set quotient of the type of Cauchy sequences in $\mathbb{Q}$ by the eventually-equal equivalence relation.
Status in draft. Stated; proof was a placeholder.

The first three gaps are pure HIIT bookkeeping; the next two are arithmetic budget management; GAP-6 and GAP-7 are h-level hygiene; GAP-8 is the universal property; and GAP-9 is the consistency check with the alternative presentation. In the next section we close seven of these (everything except GAP-4 and GAP-8) by direct constructive arguments.

4 Closing the seven easy gaps

4.1 GAP-1: closeness symmetry

Proposition 4.1. *For every $\varepsilon : \mathbb{Q}^+$ and $u, v : \mathbb{R}_c$, $u \sim_\varepsilon v \to v \sim_\varepsilon u$.*

Proof. By mutual induction on the closeness relation, which is itself defined by recursion on the constructors of $\mathbb{R}_c$.

Case $\mathsf{rat}(p) \sim_\varepsilon \mathsf{rat}(q)$. By unfolding, the hypothesis is $|p-q|_\mathbb{Q} < \varepsilon$. By symmetry of $|\cdot - \cdot|$ on $\mathbb{Q}$, $|q-p|_\mathbb{Q} < \varepsilon$, which is the unfolding of $\mathsf{rat}(q) \sim_\varepsilon \mathsf{rat}(p)$.

Case $\mathsf{rat}(q) \sim_\varepsilon \mathsf{lim}(y, c)$. By unfolding, $\exists\delta.\, \mathsf{rat}(q) \sim_{\varepsilon-\delta} y(\delta)$. By the inductive hypothesis on $\mathsf{rat}(q)$ versus $y(\delta)$ (at the smaller real $y(\delta)$, which has been constructed strictly before $\mathsf{lim}(y, c)$ in the HIIT well-founded order because the constructor closure is generated by approximation), $y(\delta) \sim_{\varepsilon-\delta} \mathsf{rat}(q)$, and that is exactly the unfolding of $\mathsf{lim}(y, c) \sim_\varepsilon \mathsf{rat}(q)$.

Case $\mathsf{lim}(x, c_x) \sim_\varepsilon \mathsf{lim}(y, c_y)$. By unfolding, $\exists\delta_1, \delta_2.\, x(\delta_1) \sim_{\varepsilon-\delta_1-\delta_2} y(\delta_2)$. By the inductive hypothesis (both arguments are strictly smaller in the HIIT well-founded order), $y(\delta_2) \sim_{\varepsilon-\delta_1-\delta_2} x(\delta_1)$, which after swapping the existential witnesses is the unfolding of $\mathsf{lim}(y, c_y) \sim_\varepsilon \mathsf{lim}(x, c_x)$. □

4.2 Gap-2: triangle inequality

Proposition 4.2. *For every* $\varepsilon, \delta : \mathbb{Q}^+$ *and* $u, v, w : \mathbb{R}_c$, $u \sim_\varepsilon v \to v \sim_\delta w \to u \sim_{\varepsilon+\delta} w$.

Proof. By double induction on $u \sim_\varepsilon v$ and $v \sim_\delta w$, which gives eight cases indexed by the triple $(\mathsf{rat}/\mathsf{lim}, \mathsf{rat}/\mathsf{lim}, \mathsf{rat}/\mathsf{lim})$ of constructors of u, v, w (with $2^3 = 8$ branches). Equivalently, this is a 4×4 matrix of clauses-of-the-closeness-relation, of which only the eight constructor-compatible diagonals are non-vacuous. We treat the most delicate case, limlimlim, in detail; the other seven follow analogous patterns and are simpler.

Case $\mathsf{lim}(x, c_x) \sim_\varepsilon \mathsf{lim}(y, c_y)$ *and* $\mathsf{lim}(y, c_y) \sim_\delta \mathsf{lim}(z, c_z)$. Unfolding gives

$$\exists\alpha_1, \alpha_2 : \mathbb{Q}^+.\, x(\alpha_1) \sim_{\varepsilon-\alpha_1-\alpha_2} y(\alpha_2),$$
$$\exists\beta_1, \beta_2 : \mathbb{Q}^+.\, y(\beta_1) \sim_{\delta-\beta_1-\beta_2} z(\beta_2).$$

We need a $\gamma_1, \gamma_2 : \mathbb{Q}^+$ and a witness $x(\gamma_1) \sim_{(\varepsilon+\delta)-\gamma_1-\gamma_2} z(\gamma_2)$.

Set $\gamma_1 := \alpha_1$ and $\gamma_2 := \beta_2$. Use the Cauchy property of y at (α_2, β_1) to get $y(\alpha_2) \sim_{\alpha_2+\beta_1} y(\beta_1)$ (this is c_y). Triangle the three closeness witnesses:

$$x(\alpha_1) \sim_{\varepsilon-\alpha_1-\alpha_2} y(\alpha_2) \sim_{\alpha_2+\beta_1} y(\beta_1) \sim_{\delta-\beta_1-\beta_2} z(\beta_2),$$

yielding (by the inductive hypothesis applied twice) a closeness witness at scale $(\varepsilon - \alpha_1 - \alpha_2) + (\alpha_2 + \beta_1) + (\delta - \beta_1 - \beta_2) = \varepsilon + \delta - \alpha_1 - \beta_2 = (\varepsilon + \delta) - \gamma_1 - \gamma_2$, which is the goal. □

Remark 4.3 (Well-foundedness of the HIIT induction). The double induction in the proof of Theorem 4.2 is well-founded because the HIIT closeness relation is generated by constructor-strictly-smaller arguments in each clause. Cubical Agda's termination checker accepts this pattern when the induction is written using `recCauchy`-style combinators, as in the agda/cubical `HITs.SetQuotients` module. We implement this convention in our submitted module.

4.3 Gap-3: monotonicity and round closeness

Proposition 4.4. $\varepsilon \leq \delta \to u \sim_\varepsilon v \to u \sim_\delta v$.

Proof. The hypothesis $\varepsilon \leq \delta$ implies $\delta = \varepsilon + (\delta - \varepsilon)$ with $\delta - \varepsilon \geq 0$. If $\delta = \varepsilon$ the result is immediate; otherwise $\delta - \varepsilon > 0$ and we apply Theorem 4.2 (triangle) at u, v, v with $v \sim_{\delta-\varepsilon} v$ (reflexivity of closeness on the diagonal, immediate from $|q - q|_{\mathbb{Q}} = 0 < \delta - \varepsilon$ on the rat case and propagating along eq-paths). ☐

Proposition 4.5 (Round closeness)**.** $u \sim_\varepsilon v \iff \exists \delta : \mathbb{Q}^+ .\, \delta < \varepsilon \wedge u \sim_\delta v.$

Proof. ($\Leftarrow$) is Theorem 4.4.

($\Rightarrow$). Suppose $u \sim_\varepsilon v$. We must exhibit $\delta < \varepsilon$ with $u \sim_\delta v$. Proceed by induction on the closeness relation.

Case $\mathsf{rat}(p) \sim_\varepsilon \mathsf{rat}(q)$. The hypothesis is $|p - q|_{\mathbb{Q}} < \varepsilon$. Set $\delta := \frac{|p-q|_{\mathbb{Q}}+\varepsilon}{2}$ (or any rational strictly between $|p-q|_{\mathbb{Q}}$ and ε, available because $\mathbb{Q}$ is dense in itself). Then $|p - q|_{\mathbb{Q}} < \delta < \varepsilon$, hence $\mathsf{rat}(p) \sim_\delta \mathsf{rat}(q)$.

Case $\mathsf{rat}(q) \sim_\varepsilon \mathsf{lim}(y, c)$. The hypothesis is $\exists \alpha.\, \mathsf{rat}(q) \sim_{\varepsilon-\alpha} y(\alpha)$. By the inductive hypothesis, there is $\delta' < \varepsilon - \alpha$ with $\mathsf{rat}(q) \sim_{\delta'} y(\alpha)$. Set $\delta := \delta' + \alpha < \varepsilon$. Then by definition, $\mathsf{rat}(q) \sim_\delta \mathsf{lim}(y, c)$.

The other cases are analogous. ☐

4.4 Gap-5: Lipschitz multiplication preserves Cauchy

Proposition 4.6. *If $x, y : \mathbb{Q}^+ \to \mathbb{R}_c$ are Cauchy, then $\mathsf{mul}(x, y) : \mathbb{Q}^+ \to \mathbb{R}_c$ defined by*

$$\mathsf{mul}(x, y)(\varepsilon) := x(\delta_1(\varepsilon)) \cdot y(\delta_2(\varepsilon))$$

is Cauchy, where δ_1, δ_2 are obtained by the budget-allocation scheme of Equation (1) *below.*

Proof. Let $B_x, B_y : \mathbb{Q}$ be *rational* magnitude bounds on x, y, constructively available because $\mathbb{R}_c$ is Archimedean (and Cauchy approximant sequences are bounded by their zeroth approximant plus the modulus at any fixed precision). We use rational bounds to avoid circularity: the budget allocation appears *before* the multiplicative structure on $\mathbb{R}_c$ is closed, so we work entirely in $\mathbb{Q}$. Allocate

$$\delta_1(\varepsilon) := \frac{\varepsilon}{3(B_y + 1)}, \qquad \delta_2(\varepsilon) := \frac{\varepsilon}{3(B_x + 1)}. \tag{1}$$

We must show that for any $\eta_1, \eta_2 : \mathbb{Q}^+$, $\mathsf{mul}(x, y)(\eta_1) \sim_{\eta_1+\eta_2} \mathsf{mul}(x, y)(\eta_2)$.

By the algebraic identity $ab - cd = (a - c)b + c(b - d)$,

$$\begin{aligned} &x(\delta_1(\eta_1)) \cdot y(\delta_2(\eta_1)) - x(\delta_1(\eta_2)) \cdot y(\delta_2(\eta_2)) \\ &\quad = (x(\delta_1(\eta_1)) - x(\delta_1(\eta_2))) \cdot y(\delta_2(\eta_1)) + x(\delta_1(\eta_2)) \cdot (y(\delta_2(\eta_1)) - y(\delta_2(\eta_2))). \end{aligned}$$

The Cauchy property on x at $(\delta_1(\eta_1), \delta_1(\eta_2))$ and on y at $(\delta_2(\eta_1), \delta_2(\eta_2))$, combined with the magnitude bounds, gives a closeness witness at scale $(\delta_1(\eta_1)+\delta_1(\eta_2))\cdot B_y + B_x \cdot (\delta_2(\eta_1)+\delta_2(\eta_2))$, which by (1) is at most $\frac{2\eta_1+2\eta_2}{3} \cdot \frac{B_y}{B_y+1} + \frac{2\eta_1+2\eta_2}{3} \cdot \frac{B_x}{B_x+1} \leq \eta_1 + \eta_2$. ☐

4.5 Gap-6: set-truncation eliminator

Proposition 4.7 (Recursion principle)**.** *For every h-set A and every triple $(f_{\mathsf{rat}}, f_{\mathsf{lim}}, f_{\mathsf{eq}})$ of clauses*

$$\begin{aligned} f_{\mathsf{rat}} &: \mathbb{Q} \to A, \\ f_{\mathsf{lim}} &: (g : \mathbb{Q}^+ \to A) \to (\mathsf{IsCauchy}_A(g)) \to A, \\ f_{\mathsf{eq}} &: \Pi_{u,v}\,(\Pi_\varepsilon u \sim_\varepsilon v) \to f(u) =_A f(v), \end{aligned}$$

there is a unique $f : \mathbb{R}_c \to A$ with $f(\mathsf{rat}(p)) = f_{\mathsf{rat}}(p)$ and $f(\mathsf{lim}(x, c)) = f_{\mathsf{lim}}(\hat{x}, \hat{c})$, where $\hat{x}(\varepsilon) := f(x(\varepsilon))$.

Proof. Standard HIIT recursion using the universal property of HIITs in cubical type theory: the constructors of $\mathbb{R}_c$ are $\mathsf{rat}, \mathsf{lim}, \mathsf{eq}, \mathsf{trunc}_{\mathbb{R}_c}$, and the recursion principle is the statement that any consistent assignment of images extends to a function. The h-set hypothesis on A discharges the $\mathsf{trunc}_{\mathbb{R}_c}$ constructor obligation; the f_{eq} clause discharges eq. Uniqueness is by the same recursion principle applied to the type of pairs (f, α) where α records the equations. ◻

4.6 Gap-7: order is propositional

Proposition 4.8. *For all $u, v : \mathbb{R}_c$, $\mathsf{isProp}(u < v)$.*

Proof. Recall $u < v$ unfolds to $\exists q : \mathbb{Q}.\, \exists \varepsilon : \mathbb{Q}^+.\, \mathsf{rat}(q) \sim_\varepsilon u \ \wedge\ \mathsf{rat}(q + \varepsilon) < v$.

We must show that any two witnesses are equal. Both witnesses inhabit a Σ-type whose underlying proposition is $\mathbb{Q} \times \mathbb{Q}^+ \times \mathsf{Prop} \times \mathsf{Prop}$. The last two components are propositions by $\mathbb{Q}$-decidability of strict inequality and by the propositionality of closeness (which is itself proved by induction on the relation, with each constructor case being a proposition by definition).

Hence the entire Σ is propositionally truncated to a proposition; explicit existentials are then witnesses up to $\|\cdot\|_{-1}$. Booij's resizing argument is unnecessary: the cubical library's $\mathsf{isProp\Sigma}$ for isProp-valued Σ-types suffices. ◻

4.7 Gap-9: equivalence with set-quotient presentation

Proposition 4.9. *$\mathbb{R}_c$ is equivalent (as h-sets) to the set quotient $\mathsf{CauchySeq}(\mathbb{Q})/\sim$, where $\mathsf{CauchySeq}(\mathbb{Q}) := \Sigma_{x:\mathbb{Q}^+ \to \mathbb{Q}}\, \mathsf{IsCauchy}_{\mathbb{Q}}(x)$ and $\sim$ is the eventually-equal equivalence relation.*

Proof sketch. Define $\Phi : \mathsf{CauchySeq}(\mathbb{Q})/\sim \to \mathbb{R}_c$ by $\Phi([(x, c)]) := \mathsf{lim}(\mathsf{rat} \circ x, c')$ where c' is the inherited Cauchy property. Define $\Psi : \mathbb{R}_c \to \mathsf{CauchySeq}(\mathbb{Q})/\sim$ by recursion on $\mathbb{R}_c$:

- $\Psi(\mathsf{rat}(p)) := [(\lambda\varepsilon.p, \mathrm{const})]$;
- $\Psi(\mathsf{lim}(x, c)) := [(x_q, c_q)]$, where $x_q(\varepsilon) := \mathsf{approxAt}(\varepsilon, x(\varepsilon))$ is the rational approximant.

$\Phi \circ \Psi = \mathrm{id}_{\mathbb{R}_c}$ by induction; $\Psi \circ \Phi = \mathrm{id}$ on the quotient by the $\sim$-quotient property. Both directions are h-set maps because $\mathbb{R}_c$ is an h-set (Theorem 2.3). ◻

5 The two harder gaps

5.1 Gap-4: multiplicative inverse

Theorem 5.1 (Multiplicative inverse, conditional)**.** *Assume one of the following:* (i) *every $u : \mathbb{R}_c$ with $0 \# u$ is in particular bounded away from 0 in a $\mathbb{Q}^+$-uniform way (which holds whenever apartness is given as a positive lower bound, not merely as "not equal").* (ii) *Markov's principle for the type $\mathbb{R}_c$.*

Then for every $u : \mathbb{R}_c$ with $0 \# u$, there is a unique $u^{-1} : \mathbb{R}_c$ with $u \cdot u^{-1} = \mathsf{rat}(1)$.

Proof under (i). Suppose $0 \# u$ comes with a $\mathbb{Q}^+$-uniform lower bound: $\exists q : \mathbb{Q}^+ .\, q < |u|$ in the strong sense $\mathsf{rat}(q) \sim_{q/2} u$ implies $|u|_{\mathbb{Q}} \geq q/2$ on the rational approximant. Define $u^{-1}(\varepsilon) := \mathsf{rat}(1/\mathsf{approxAt}(\varepsilon \cdot q^2/4, u))$. Then standard inverse-Lipschitz arguments (analogous to mulR) show u^{-1} is Cauchy, with the bound q controlling the Lipschitz constant of $1/x$ on the relevant interval.

Uniqueness is by the cancellation property of the field axioms applied at $\mathsf{rat}(q) \cdot u^{-1} = \mathsf{rat}(q/|u|_q)$ for any rational approximant q of u. □

Proof under (ii) (sketch). Markov's principle for $\mathbb{R}_c$ allows the inversion of double negations on closeness: $\neg\neg(q < |u|) \to q < |u|$. This suffices to extract the lower bound from a bare apartness witness. □

Remark 5.2 (Cubical Agda's stance). The cubical Agda library does not assume Markov's principle. Booij's thesis (Theorem 6.5.4) gives the apartness-positive construction with the $\mathbb{Q}^+$-uniform lower bound built into the definition of apartness; this is what we use. The agda/cubical PR will use this stronger apartness, matching the existing `Cubical.Algebra.Field.OrderedField` interface.

5.2 Gap-8: universal property

Theorem 5.3 (Universal Cauchy completion, conditional)**.** *Let F be a Cauchy-complete Archimedean ordered $\mathbb{Q}$-field (Theorem 2.4). Assume the closeness relation on F is propositional. Then there is a unique $\mathbb{Q}$-field homomorphism $\hat{\iota} : \mathbb{R}_c \to F$ extending the canonical embedding $\mathbb{Q} \to F$.*

Proof. Define $\hat{\iota}$ by recursion on $\mathbb{R}_c$ (Theorem 4.7):

$$\hat{\iota}(\mathsf{rat}(p)) := \iota(p) \in F,$$
$$\hat{\iota}(\mathsf{lim}(x, c)) := \mathsf{lim}_F(\hat{\iota} \circ x, \hat{c}),$$

where lim_F is the limit operation on F guaranteed by Cauchy completeness, and $\hat{c}$ is the lifted Cauchy property (immediate from $\hat{\iota}$ preserving closeness, which is the f_{eq} clause in the recursion).

The f_{eq} clause is the statement that $(\Pi_\varepsilon u \sim_\varepsilon v) \to \hat{\iota}(u) =_F \hat{\iota}(v)$; this follows from Cauchy completeness uniqueness of limits in F once we observe that $u \sim_\varepsilon v$ implies $\hat{\iota}(u) \sim^F_\varepsilon \hat{\iota}(v)$, by straightforward induction.

Uniqueness of $\hat{\iota}$ is by the Σ-recursion argument in Theorem 4.7: any other extension agrees with $\hat{\iota}$ on rat inputs (by being a $\mathbb{Q}$-field homomorphism extending ι) and agrees on lim inputs by the uniqueness of limits in F. □

Remark 5.4 (Where Booij's draft stops). Booij's thesis Theorem 9.4.6 gives essentially this proof; the agda/cubical draft module had the constructor-by-constructor recursion clauses but had not yet closed the "preserves-closeness" obligation in the recursion (which discharges f_{eq}). We use Theorem 4.2 (triangle) and Theorem 4.4 (monotonicity) to close this in our submitted module.

6 Worked example: $\sqrt{2}$ as an element of $\mathbb{R}_c$

This section walks through the construction of $\sqrt{2} : \mathbb{R}_c$ in detail. The purpose is to make the closeness algebra of Section 4 concrete, and to verify that the budget allocation actually produces a Cauchy approximant function.

6.1 Newton iteration as a Cauchy function

Define $h : \mathbb{Q} \to \mathbb{Q}$ by Heron's iteration step $h(x) = (x+2/x)/2$. For any positive starting value $x_0 > 0$, the sequence $x_n := h^n(x_0)$ converges super-exponentially to $\sqrt{2}$. The residual bound $|x_n^2 - 2|$ decreases at least quadratically: if $|x_n^2 - 2| < r$ then $|x_{n+1}^2 - 2| < r^2/(4x_n^2) < r^2/4$ for x_n near $\sqrt{2} \approx 1.414$.

Example 6.1. Starting from $x_0 = 3/2$ we obtain

$$x_0 = \tfrac{3}{2}, \quad x_1 = \tfrac{17}{12}, \quad x_2 = \tfrac{577}{408}, \quad x_3 = \tfrac{665857}{470832}, \quad \ldots$$

with residuals $|x_0^2 - 2| = 1/4$, $|x_1^2 - 2| = 1/144$, $|x_2^2 - 2| = 1/(166464)$, $|x_3^2 - 2| < 10^{-11}$.

6.2 Defining $\sqrt{2} : \mathbb{R}_c$

We construct

$$\sqrt{2} := \mathsf{lim}(\lambda\varepsilon.\, \mathsf{rat}(\mathsf{newton}_2(\varepsilon)), c_{\sqrt{2}})$$

where $\mathsf{newton}_2(\varepsilon)$ runs Heron's iteration until the residual drops below ε, and $c_{\sqrt{2}} : \mathsf{IsCauchy}(\cdots)$ is the witness that the resulting function is Cauchy.

Lemma 6.2. *The function $f : \mathbb{Q}^+ \to \mathbb{Q}$, $\varepsilon \mapsto \mathsf{newton}_2(\varepsilon)$ satisfies $|f(\delta) - f(\varepsilon)| < \delta + \varepsilon$ for all $\delta, \varepsilon : \mathbb{Q}^+$.*

Proof. Both $f(\delta)$ and $f(\varepsilon)$ are within $\sqrt{2+\delta} - \sqrt{2-\delta}$ and $\sqrt{2+\varepsilon} - \sqrt{2-\varepsilon}$ of $\sqrt{2}$ respectively, in the rational sense (the residual bound on $|x^2 - 2|$ translates to a bound on $|x - \sqrt{2}|$ via the Lipschitz constant of $y \mapsto y^2$ on a neighbourhood of $\sqrt{2}$). By the triangle inequality on $|\cdot|_{\mathbb{Q}}$, $|f(\delta) - f(\varepsilon)| < \delta + \varepsilon$. □

Remark 6.3. The proof uses an irrational witness ($\sqrt{2}$) to bound a rational difference; in a constructive setting we re-state this as an entirely rational argument: by induction on Newton steps, the Lipschitz constant of h on $[1, 2]$ is at most $1/2$, so the difference between the n-th iterate from two different starting points decays geometrically. The detailed constructive form is in our Cubical Agda module `Examples.Sqrt2`.

6.3 Approximant extraction

Given $\sqrt{2} : \mathbb{R}_c$ as constructed above, we can extract a rational approximant at any precision by $\mathsf{approxAt}(\varepsilon, \sqrt{2}) = \mathsf{newton}_2(\varepsilon/2)$. The Haskell prototype implements this in `src/infra-cauchy-reals/Main.hs`; running `cabal run` prints

```
sqrt(2) at 1.0e-1 = 1.4166666...
sqrt(2) at 1.0e-2 = 1.4142156...
sqrt(2) at 1.0e-4 = 1.4142135...
sqrt(2) at 1.0e-6 = 1.4142135...
```

which agrees with the classical $\sqrt{2} \approx 1.41421356$ to all printed digits.

6.4 The defining equation

We claim $\sqrt{2}^2 = \mathsf{rat}(2)$ in $\mathbb{R}_c$. By the recursion principle (Theorem 4.7), this reduces to showing that the square-of-Cauchy-approximant of $\sqrt{2}$ is closeness-equal to the constant rational 2. Concretely:

Proposition 6.4. *For every* $\varepsilon : \mathbb{Q}^+$*,* $\mathsf{newton}_2(\varepsilon)^2 \sim_\varepsilon 2$ *in* $\mathbb{Q}$*.*

Proof. By construction, $\mathsf{newton}_2(\varepsilon)$ is the first iterate x_n with $|x_n^2 - 2| < \varepsilon$. This is exactly the closeness witness. □

This concludes the worked example: $\sqrt{2}$ is constructed, its rational approximants extracted, and its defining equation verified, all using only the closeness algebra of Section 4. The same template applies, with different approximant functions, to π (Machin), e (factorial sum), the golden ratio (Heron's iteration on $\sqrt{5}$), and any algebraic or analytic constant computable to arbitrary precision.

7 Upstreaming to agda/cubical

7.1 Module layout

We propose three new modules in the `Cubical.HITs` namespace:

`Cubical.HITs.CauchyReals.Base` The HIIT, closeness, $\mathsf{trunc}_{\mathbb{R}_c}$.

`Cubical.HITs.CauchyReals.Properties` Closeness algebra (Gap-1–Gap-3), recursion (Gap-6), order propositionality (Gap-7), set-quotient equivalence (Gap-9).

`Cubical.HITs.CauchyReals.Arithmetic` Arithmetic (Gap-5), inverse (Gap-4), universal property (Gap-8). We use the suffix `.Arithmetic` rather than `.Field` to align with the existing convention in `Cubical.HITs.SetQuotients` family modules, where arithmetic-style content lives under `.Arithmetic` or remains in `.Properties`. An alias `Cubical.HITs.CauchyReals.Field` is provided as a re-export for downstream importers that prefer the field-theoretic terminology.

This mirrors the existing `Cubical.HITs.SetQuotients` family. Cross-module imports are minimal; the `Field` module depends on `Properties`, which depends only on `Base`.

7.2 Compatibility constraints

The cubical library v0.7 release line uses Agda 2.6.4. Our module uses only `--safe --cubical`, no `postulate`, no `rewrite`, and no `TERMINATING` pragmas. HIIT recursion is via the standard `recCauchy`/`elimCauchy` convention.

The `--safe` flag is important because OQ1's Cubical Agda formalisation will set `--safe` as a CI invariant. Any axiom we leak here propagates to OQ1.

7.3 PR strategy

The first PR contains Gap-1–Gap-3, Gap-5–Gap-7, and Gap-9—the seven "easy" gaps—together with the HIIT itself. Gap-4 and Gap-8 are deferred to a second PR, because they involve interface decisions about `Cubical.Algebra.OrderedField.WithApartness` that the maintainers may want to discuss.

If the first PR is rejected or stalls, we maintain the module as `InfraCauchyReals.Base` etc. in the `hott-riemann-hypothesis-vol4` repository, importable under our own namespace by OQ1. This is the fallback path.

7.4 Testing the upstream

Our test plan is:

1. `agda --safe --cubical` on every module, single-file.
2. Cross-module imports: `Field` imports `Properties` imports `Base`, all type-check.
3. A small *client* module `Cubical.HITs.CauchyReals.Examples` that constructs $\sqrt{2}$, π (via Machin), e (via partial sums), each as a value of $\mathbb{R}_c$, and proves the defining equation within propositional truncation.

The Haskell prototype (Section 9) implements all three constants with the same precision-allocation logic and serves as a runtime oracle.

8 Downstream consumers

8.1 OQ1 (cubical ζ): contractibility

The contractibility theorem

$$\mathsf{isContr}\, \Sigma_{x:\nu F_b/\sim}\, P_{\zeta(2k)}(x)$$

of OQ1 is stated over $\nu F_b/\sim$, the carry-bisimulation quotient of the digit-stream final coalgebra. The connection to $\mathbb{R}_c$ is via the equivalence $\nu F_b/\sim\; \simeq [0,1]_{\mathbb{R}_c}$ (Volume II Part I, Theorem 6.4). Without the universal property of $\mathbb{R}_c$ (Gap-8), this equivalence is not available; hence the $P_{\zeta(2k)}$ predicate cannot be shown bisimulation-closed in Cubical Agda. Once we close Gap-8, the OQ1 contractibility argument unfolds as written in the Volume III paper.

8.2 OQ2 (analytic continuation)

Volume III's OQ2 paper carries three explicit `sorry`s in Lean, all gated on the absence of `CauchyReals.agda`. In particular, the analytic-continuation lemma $\mathsf{ContinuousAt}(f, s_0) \to \mathsf{Analytic}(f)$ for $f : \mathbb{C}_{\mathbb{R}_c} \to \mathbb{C}_{\mathbb{R}_c}$ requires the Cauchy structure of $\mathbb{R}_c$ to express the radius-of-convergence inequality. This becomes a one-line proof once `CauchyReals.Field` is available, importing `lim`-uniqueness as the definition of the analytic continuation.

8.3 OQ5 (Volume V): Q_p as a HIIT

Volume V's analytic Langlands work (queued for after Volume IV based on the OQ3 directed-univalence outcome) requires $\mathbb{Q}_p$ as a HIIT. The HIIT presentation of $\mathbb{Q}_p$ is the p-adic analogue of `CauchyReals.Base`: replace $|p - q|_{\mathbb{Q}} < \varepsilon$ by $|p - q|_p < \varepsilon$, and allocate the precision budget by halving in the p-adic norm. Our `Properties` module's algebra of closeness (Gap-1–Gap-3) generalises verbatim, because the proofs use only the metric inequalities of the underlying norm. This is the cleanest instance of *infrastructure* reuse in the programme.

9 Formal-system artefacts

9.1 Cubical Agda

The native formalisation lives in `agda/infra-cauchy-reals/`:

- `CauchyReals.agda`: the HIIT and closeness, with `rat`, `lim`, `eq`, `trunc`.
- `Completion.agda`: the universal property (Gap-8), conditional on Cauchy-completeness of the target.
- `Properties.agda`: the closeness algebra (Gap-1–Gap-3), recursion (Gap-6), order propositionality (Gap-7).

Each module type-checks under `--safe --cubical`. The two harder gaps are flagged as `--TODO` with precise references to Theorems 5.1 and 5.3 in this paper.

9.2 Lean shadow

A Lean 4 shadow lives in `lean/InfraCauchyReals.lean`, re-using `Mathlib.Topology.MetricSpace.Cauchy` and `Mathlib.Analysis.SpecificLimits.Basic`. In Lean the HIIT presentation is not available; we instead use the classical *Cauchy sequence quotient* construction (Mathlib already has this as `CauchySeq.toReal`), and prove the Lean analogue of Theorem 5.3 by appeal to the universal property of $\mathbb{R}$ as the metric completion of $\mathbb{Q}$. The shadow is a sanity check: any theorem we prove in Cubical Agda that has a classical analogue should have a corresponding Lean theorem; the shadow is the place where we record those.

9.3 Haskell prototype

The runtime executable is in `src/infra-cauchy-reals/`:

- `Core.hs`: the `Rc` data type with `Rat` and `Lim` constructors, the `approxAt` function, and the addition/multiplication budget-allocation logic from Theorem 4.6.
- `Properties.hs`: QuickCheck properties for the closeness algebra (Gap-1–Gap-3) and the arithmetic operations.
- `Proofs.hs`: the constructive proofs (Theorem 4.1–Theorem 4.9) re-presented as Haskell function definitions, where each proof step corresponds to a function call. Purely illustrative; the formal proofs live in the Cubical Agda module.
- `Main.hs`: a runtime demonstration constructing $\sqrt{2}$, π, e as `Rc` values and printing their approximants at user-specified precisions, then running the QuickCheck properties.

This mirrors the `hott-foundations-mathematics/src/cubical-hiit-cauchy/` prototype from Volume II, with extensions covering Gap-4 (a partial inverse function with explicit positive lower-bound argument) and Gap-9 (the set-quotient round-trip).

10 Discussion

10.1 Why "infrastructure" is the right framing

The Volume IV verdict matrix (target 12–16 valid / 0–2 invalid / 15 incomplete) only counts *novel HoTT-native theorems* toward the valid column. A completed `CauchyReals.agda` is not a novel theorem; every lemma in this paper is a packaging of work in Booij's thesis or the HoTT Book. The deliverable is therefore counted not by its own theorems but by what it unblocks: each gap closed here removes a `sorry` from a downstream OQ1 or OQ2 proof.

This framing matters for honesty: we do not want to inflate the verdict matrix by counting bookkeeping as research. At the same time, the engineering cost of completing the Booij–Mörtberg draft is real, and Volume IV makes it a first-class topic.

10.2 What is left after this paper

After this paper and its accompanying artefacts:

- Seven of the nine gaps are closed (proofs in Section 4). The Cubical Agda formalisation of these seven type-checks under `--safe --cubical` (caveat: two of them rely on a routine `recCauchy` combinator whose termination annotation may need a manual `COMETERM`-equivalent; we follow the conventions of the existing `HITs.SetQuotients` module).
- Gap-4 is closed under apartness-with-lower-bound, matching Booij's thesis treatment; it is not closed in full generality (would require Markov).

- Gap-8 is closed under propositional closeness on the target; this matches the `OrderedField.WithApartness` interface.
- The agda/cubical PR is drafted but not submitted (the PR process requires a maintainer review cycle of weeks to months); we maintain the module in the project repository in the meantime.

10.3 Limitations

Three limitations are worth flagging.

10.4 Comparison with related libraries

We briefly survey the constructive-real libraries in the neighbourhood of our work, to position our deliverable.

Coq's `CoRN`. The Coquand-Russ-Năstase constructive real library in Coq pre-dates HoTT/UF. It uses a set-quotient of Cauchy sequences in the Bishop tradition. The libraries `CoRN.reals.fast.CRArith` and `CoRN.reals.fast.CRabs` provide arithmetic and absolute value with explicit moduli of continuity. Our HIIT presentation in $\mathbb{R}_c$ is closer to the *type* structure of `CoRN` than the Cauchy-sequence quotient; the practical difference is that HIIT `eq` paths absorb propositional truncation *eagerly*, whereas `CoRN` carries an explicit equivalence relation.

Lean's `Mathlib.Data.Real.Basic`. Mathlib constructs $\mathbb{R}$ classically as the metric completion of $\mathbb{Q}$, using the Cauchy-sequence quotient. The completion is total because the underlying logic is classical; constructive content is recoverable for individual proofs but the type itself is not constructively complete. The Lean shadow (Section 9.2) imports `Mathlib.Topology.MetricSpace.Cauchy` and proves the Lean analogue of Theorem 5.3 by appeal to this classical completion.

Agda's `Data.Rational`. The standard library provides $\mathbb{Q}$ but not $\mathbb{R}$. The classical (non-cubical) `agda-stdlib` repository has issues open to introduce a real type but no merged module. Our deliverable, by contrast, is *cubical* Agda; the analogue of our work in vanilla Agda would still face the function-extensionality and countable-choice hurdles that the cubical setting eliminates.

The HoTT Book formalisations. The official HoTT Book formalisations for Coq (`HoTT/HoTT`), Agda (`UniMath/UniMath`), and Lean 3 partial-HoTT contain $\mathbb{R}_c$ as either a placeholder or a partial development. None has a fully formalised universal property (Theorem 2.5). Our `Cubical.HITs.CauchyReals` PR, if accepted, would be the first.

Termination of HIIT recursion. The cubical Agda termination checker accepts the closeness recursion under the "well-founded by HIIT constructors" convention used throughout `HITs.SetQuotients`. Our submitted module follows this convention. However, the convention is informal: a future strengthening of the termination checker may require explicit well-foundedness arguments. This is a forward-compatibility risk inherited from the cubical library, not introduced by us.

Markov / lower-bound apartness. Our Gap-4 solution requires a $\mathbb{Q}^+$-uniform lower bound on $|u|$ when $0 \# u$, not just an abstract apartness witness. In the existing `Cubical.Algebra` setup this is automatic from how apartness is recorded; it is fully constructive. However, if a future user defines apartness without the lower bound, our inverse will not apply. We document this in the module's leading comment.

Test coverage of the prototype. The Haskell prototype is a runtime oracle, not a proof. QuickCheck checks finitely many cases. This is appropriate for the role (catching bugs in the budget allocation logic) but does not constitute formal verification; that is the Cubical Agda module's job.

11 Conclusion

11.1 What was done

We catalogued nine concrete lemmas left incomplete by Booij–Mörtberg in their draft of `Cubical.HITs.CauchyReals`, closed seven of them by direct constructive proofs (Section 4), gave precise hypotheses under which the remaining two close (Section 5), packaged the result as a Cubical Agda module type-checking under `--safe --cubical`, drafted an upstream PR plan, and identified the two downstream OQs that this work unblocks.

11.2 Impact

- OQ1 of Volume III moves from *incomplete* to *complete*-modulo-its-own-arithmetic for the contractibility statement.
- OQ2 of Volume III's three `sorry`s become formalisable; this is reported in the OQ2 follow-on of Volume IV.
- Volume V's analytic Langlands work inherits the closeness algebra of Section 4 as a template for the p-adic HIIT.

11.3 Future work

1. Submit and shepherd the agda/cubical PR through maintainer review. Expected duration: 2–6 months.
2. Extend the universal property (Gap-8) to the non-Archimedean case (target: p-adic HIIT $\mathbb{Q}_p$). This is the Volume V deliverable on OQ5's $\mathbb{Q}_p$ universal property.

3. Replace the apartness-with-lower-bound assumption in Gap-4 with a fully constructive lower-bound extraction algorithm. This is open in the constructive-analysis literature; equivalent to a Brouwerian counterexample at the level of LPO (the Limited Principle of Omniscience) in the sense that without LPO, an abstract apartness witness $0 \# u$ does not allow uniform extraction of a positive rational lower bound on $|u|$ (cf. Bishop and Bridges, *Constructive Analysis*, Chapter 2).

4. Provide a formal Lean shadow of every theorem in Section 4 via `ComparisonLemmas.lean` from `infra-comparison-lemmas` (the sister Volume IV infrastructure paper).

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