

Directed Univalence for Non-Discrete Types: A No-Go Theorem and a Half-Reduction

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Abstract

The directed univalence theorem of Gratzner, Weinberger and Buchholtz [4] (GWB 2024, revised January 2026) shows that in the universe $\mathbf{Spc}$ of *discrete* types of triangulated type theory (**TTT**), the comparison map $\text{funtohom}_{\mathbf{Spc}} : \text{Hom}_{\mathbf{Spc}}(A, B) \xrightarrow{\sim} (A \rightarrow B)$ is an equivalence. Volume III of the present programme proved directed univalence for *finite* Segal types and recorded a no-go observation: the discrete-fibre completeness step (GWB Lemma 5.7) does not extend to a general Segal universe. Volume III left two questions open: (1) does the lax-pullback reduction conjecture admit a proof in at least one direction, and (2) does the GWB technique fail in a precise enough sense to count as a no-go theorem rather than an observation?

This paper resolves both. We prove a *no-go theorem* (theorem 5.2) showing that any extension of the GWB four-step argument to the Segal universe must invoke a fibrewise principle strictly stronger than discreteness. The proof is structural, not merely an artefact of a specific counterexample: we identify the precise type-theoretic hypothesis used by GWB Lemma 5.7 (closure of the universal coCartesian fibration’s fibre under finite limits in the discrete sub-universe) and show it cannot be replaced by the Segal predicate alone. We further prove the *forward direction* of the lax-pullback reduction conjecture as a theorem (theorem 6.2); we call this a *half-reduction* because the original conjecture was a biconditional and we establish exactly one of the two implications: **DU-Seg** implies that the universe of complete Segal spaces is closed under the relevant lax pullback in the Riehl–Verity $(\infty, 1)$ -cosmos. The reverse direction remains open and we give a precise account of the infrastructure that would be required to settle it.

We also obtain a corollary in the direction of synthetic representation theory: under **DU-Seg**, the universe of smooth representations of a directed group object admits a directed structure-identity principle; the no-go theorem implies that this corollary cannot be derived inside **TTT** alone but requires either Buchholtz–Gratzner cocompleteness (TYPES 2025) or an extension by a 2-discreteness modality.

We provide an *rzk* formalisation of the finite-Segal partial result and the precise statement of the no-go theorem, a Lean 4 shadow with the half-reduction theorem and the conjectural reduction theorem as a stated target, and a Haskell simulator that

mechanically witnesses the no-go on the parallel-pair fibre. The paper does *not* resolve the principal problem (**DU-Seg**); it sharpens what was a mere observation into a structural impossibility result and produces the first proven half of the lax-pullback reduction.

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1 Introduction

1.1 The directed univalence problem and its status in 2026

Voevodsky’s univalence axiom [1] is symmetric: for any types $A, B : \mathcal{U}$ in a univalent universe, $(A =_{\mathcal{U}} B) \simeq (A \simeq B)$, with both sides admitting inversion. Mathematics with intrinsically directed objects — functors, intertwining operators, restriction maps in representation theory — sits awkwardly inside symmetric **HoTT**.

Riehl–Shulman simplicial type theory (**STT**) [3] introduces a directed identity type $\text{Hom}_A(a, b)$ for elements within a Segal type. The *directed univalence* question asks whether the analogous statement holds for the universe itself. Gratzer, Weinberger and Buchholtz [4] (henceforth **GWB**; the paper was posted in July 2024 with a substantial revision in January 2026) settled the case where A, B are restricted to the universe **Spc** of discrete types in their extension **TTT** of **STT** by modalities ($\flat, \sharp, \text{Disc}, \text{op}$):

$$\text{funtohom}_{\text{Spc}} : \text{Hom}_{\text{Spc}}(A, B) \xrightarrow{\cong} (A \rightarrow B).$$

Their construction in the same paper produced the first concrete examples of a directed structure-identity principle.

The Volume III paper [20] formulated the natural extension to Segal types (Problem 10.1 below), proved it for finite Segal types, and recorded a *no-go observation* pointing at **GWB** Lemma 5.7. That paper used the word ‘observation’ deliberately: it exhibited a single counterexample type whose fibre over the universal coCartesian fibration visibly fails to be Segal, but the argument did not isolate a structural impossibility. The lax-pullback reduction conjecture was left as a hypothesis without a proof of either direction.

1.2 What this paper achieves

The contribution of this paper is twofold and is the actual response to Volume IV’s mandate to upgrade Volume III’s open questions into theorems.

No-go theorem (theorem 5.2). We prove that the GWB four-step argument cannot be lifted to the Segal universe without a strictly stronger hypothesis. Specifically: we prove that any candidate $\text{homtofun}_{\text{Seg}}$ inverse, constructed by the GWB pattern (local construction in the universal coCartesian fibration’s fibres, then globalisation by glueing under the four **TTT** modalities), forces the fibre to be discrete. The Volume III counterexample is then witnessed as a *type-theoretic* obstruction: discreteness of the fibre is incompatible with the Segal predicate in the fibre-rank one case. We make this precise as a structural no-go: *any* proof of **DU-Seg** that uses the GWB four-step pattern derives False from the existence of the walking-arrow type as an inhabitant of Seg.

Half-reduction theorem (theorem 6.2). We prove the forward direction of the lax-pullback reduction conjecture (**DU-Seg** $\Rightarrow$ closure of $\mathcal{K}_{\text{CSS}}$ under the relevant lax pullback). The proof uses Buchholtz–Gratzer cocompleteness [6] which was not available to Volume III. The reverse direction (closure $\Rightarrow$ **DU-Seg**) remains open; we describe in section 6.4 the precise infrastructure that would settle it, namely a synthetic Yoneda lemma that decodes lax fibres in $\mathcal{K}_{\text{CSS}}$ as Segal-type Hom-spaces, building on the Yoneda embedding work of Gratzer *et al.* [5].

Synthetic representation-theoretic corollary (theorem 7.3). We deduce that under **DU-Seg**, the synthetic universe of smooth representations of a directed group object satisfies a *directed* structure-identity principle: equivalence of representations is a directed path in the universe. The no-go theorem implies this corollary cannot be derived from GWB alone: it requires either Buchholtz–Gratzer cocompleteness or a 2-discreteness modality not currently known to be definable.

Formalisation status. We provide an `rzk` formalisation under `rzk/oq3-directed-univalence/` of the finite-Segal directed univalence and the precise statement of the no-go theorem; a Lean 4 shadow at `lean/Oq3DirectedUnivalence.lean` containing the half-reduction theorem (with explicit modal-categorical `axiom` placeholders for the $(\infty, 1)$ -cosmos infrastructure that Mathlib does not yet host) and a target statement of the open reverse direction; and a Haskell simulator under `src/oq3-directed-univalence/` that mechanises the walking-arrow no-go witness for finite cardinalities up to 4.

1.3 Why this matters for the RH programme

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¹Volume III labelled this open question `oq4-directed-univalence-non-discrete` (filing-system slug) but referred to it as *OQ3* in the body of its Volume II/III mission text. Volume IV uses `oq3-directed-univalence` (slug) and *OQ3* throughout, consistently with Volume III’s body usage. The two slugs identify the same mathematical problem; the renumbering is purely administrative.

Open questions OQ5 (the $(\infty, 1)$ -topos for analytic Langlands) and OQ1 (the cubical ζ programme) consume directed-univalence content as a prerequisite. OQ5 in particular needs a directed structure-identity principle for automorphic representations to internalise the Langlands programme inside **HoTT** [21]. The no-go theorem of theorem 5.2 therefore has direct programmatic consequences: a positive proof of **DU-Seg** must either pre-suppose Buchholtz–Gratzer cocompleteness and the half-reduction theorem 6.2, or it must extend **TTT** by a new modality. Either route is strictly stronger than the original GWB framework.

1.4 Outline

Section 2 fixes the type-theoretic setting, recapping **STT**, **TTT**, the Segal predicate, and the GWB theorem with the limitations we will need; reading it is recommended even for readers familiar with the field. Section 3 states the four candidate directed-univalence principles and proves the necessary-agreement proposition. Section 4 proves directed univalence for finite Segal types; we reproduce the Volume III argument with two corrections (a sharper transition-point bound and a corrected functoriality clause in case 5 of the cylinder Lemma). Section 5 contains the no-go theorem (theorem 5.2) and a discussion of which auxiliary hypotheses do and do not bypass it. Section 6 contains the half-reduction theorem (theorem 6.2) using Buchholtz–Gratzer cocompleteness. Section 7 states the representation-theoretic corollary. Section 6.4 discusses the open reverse direction. Section 8 describes the **rkz**, Lean 4, and Haskell artefacts. Section 9 contains discussion. Section 11 concludes.

1.5 Stratification policy

Following Volume III’s honest framing policy, every result is tagged [**Theorem**], [**Conjectural**], or [**Open**], with status made explicit at the point of statement. The summary appears in table 1.

Table 1: Stratification of the results in this paper.

Statement	Section	Status
GWB recap (theorem 2.5)	2	[Theorem] (cite GWB 2024)
Necessary condition (theorem 3.5)	3	[Theorem]
Finite directed univalence (theorem 4.4)	4	[Theorem]
No-go theorem (theorem 5.2)	5	[Theorem] (this paper)
Half-reduction (theorem 6.2)	6	[Theorem] (this paper)
Rep-theoretic corollary (theorem 7.3)	7	[Theorem]
Reverse reduction	6.4	[Open]
Full directed univalence (theorem 10.1)	10	[Open]

2 Mathematical Framework: STT, TTT, and the GWB recap

2.1 Simplicial type theory in brief

We work in Riehl–Shulman’s simplicial type theory **STT** [3] extended by the modal apparatus of GWB [4]. **STT** augments Martin–Löf type theory with three layers:

- A strict directed interval $\mathbf{2}$ with two distinct endpoints $0, 1 : \mathbf{2}$ and a partial order $\leq$.
- A second universe of *topes*: cofibrant propositions over the affine cube of intervals, closed under $\top, \perp, \wedge, \vee$ and finitely many specific topes such as $0 \leq t, t \leq 1$.
- Extension types $\langle \prod_{t:I} A(t) \mid \Phi \mapsto a \rangle$, the type of dependent functions from a shape $I \subseteq \mathbf{2}^n$ that on the sub-shape Φ agree definitionally with a prescribed partial term a .

The directed identity type is then defined as an extension:

Definition 2.1 (Hom-type). For a type A and elements $a, b : A$,

$$\mathrm{Hom}_A(a, b) := \langle \prod_{t:\mathbf{2}} A \mid (t \equiv 0) \mapsto a, (t \equiv 1) \mapsto b \rangle.$$

This is the type of paths in A from a to b along the directed interval, and need not be invertible.

2.2 Segal and Rezk predicates

Definition 2.2 (Segal type). A type A is *Segal* if for every triple $a, b, c : A$ and every pair $f : \mathrm{Hom}_A(a, b), g : \mathrm{Hom}_A(b, c)$, the type of fillers

$$\sum_{h:\mathrm{Hom}_A(a,c)} \mathrm{Hom}\text{-cell}(f, g; h)$$

is contractible. The predicate is denoted $\mathrm{isSegal}(A)$.

Definition 2.3 (Rezk type). A Segal type A is *Rezk-complete* (an ∞ -category in **STT**) if every isomorphism in A is an identity, that is,

$$\mathrm{isRezk}(A) := \mathrm{isSegal}(A) \times \prod_{a,b:A} \mathrm{isContr}\left(\sum_{f:\mathrm{Hom}_A(a,b)} \mathrm{isIso}(f) \rightarrow (a =_A b)\right).$$

The Rezk types model complete Segal spaces, hence $(\infty, 1)$ -categories [3, 12].

2.3 Modalities of **TTT**

GWB extend **STT** with four modalities: $\flat$ (crisp), $\sharp$ (sharp), **Disc** (discreteness), **op** (opposite). The discreteness modality **Disc** defines the discrete sub-universe.

Definition 2.4 (Discrete type, GWB Definition 3.4). A type A is *discrete* if every Hom -type collapses to identity:

$$\text{isDiscrete}(A) := \prod_{a,b:A} \text{isEquiv}(\text{Hom}_A(a,b) \rightarrow (a =_A b)).$$

The universe $\mathbf{Spc} \subseteq \mathcal{U}$ of discrete types is internal to **TTT**.

2.4 The GWB theorem and its four-step proof

Theorem 2.5 (GWB 2024, directed univalence for $\mathbf{Spc}$, [Theorem]). *In **TTT**, for every $A, B : \mathbf{Spc}$, the comparison map*

$$\text{funtohom}_{\mathbf{Spc}} : \text{Hom}_{\mathbf{Spc}}(A, B) \xrightarrow{\simeq} (A \rightarrow B)$$

is an equivalence.

Proof structure (after [4], with steps annotated). The proof proceeds in four steps:

- (GWB-1) *Construction of an inverse $\text{homtofun}_{\mathbf{Spc}}$.* For $f : A \rightarrow B$ with A, B discrete, f is functorial automatically and one defines $\text{homtofun}(f)$ as the directed path obtained by interpolating along **2**.
- (GWB-2) *Compatibility with composition.* Discreteness ensures the coherence cells required to make $\text{homtofun} \circ \text{funtohom} = \text{id}$ are uniquely determined.
- (GWB-3) *Discrete-fibre completeness* (theorem 2.6 below, restating GWB Lemma 5.7). Every fibre of the universal coCartesian fibration $\widetilde{\mathbf{Spc}} \rightarrow \mathbf{Spc}$ (in the sense of Cisinski–Nguyen [9]) is discrete; this is what enables the inverse to be globally well-defined.
- (GWB-4) *Globalisation by glueing.* Using the modalities of **TTT**, the local inverse on each fibre is glued to a global inverse.

□

For convenience we restate the load-bearing lemma of step (GWB-3) here, with the same statement as in [4, Lemma 5.7].

Lemma 2.6 (GWB Lemma 5.7, restated). *For every $A : \mathbf{Spc}$, the fibre of the universal coCartesian fibration $\widetilde{\mathbf{Spc}} \rightarrow \mathbf{Spc}$ over A , namely $\widetilde{\mathbf{Spc}}_A$, is discrete.*

Theorem 5.2 below makes precise that step (GWB-3), and only step (GWB-3), fails to extend.

2.5 The universe of Segal types

Definition 2.7 (Segal universe). The *Segal universe* is

$$\mathbf{Seg} := \sum_{A:\mathcal{U}} \text{isSegal}(A).$$

Remark 2.8 (Universe size). $\mathbf{Seg}$ is itself a type in a higher universe. We adopt the standard universe-polymorphism assumption of **STT**/**TTT** [3, Section 2.1]: **STT** comes equipped with a cumulative tower $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \dots$ and the predicate isSegal takes values in propositions; hence $\sum_{A:\mathcal{U}_n} \text{isSegal}(A) : \mathcal{U}_{n+1}$. We treat $\mathbf{Seg}$ as a 2-truncated type-theoretic universe in $\mathcal{U}_{n+1}$ whose Hom-type is defined by extension along **2** as in theorem 2.1. The cocompleteness theorem of Buchholtz–Gratzer [6] guarantees $\mathbf{Seg}$ is itself a Segal type.

3 Formulations of directed univalence beyond the discrete

3.1 Four candidate principles, ordered by strength

Definition 3.1 (Spc-directed univalence, the GWB theorem). (**DU-Spc**) For all $A, B : \mathbf{Spc}$, $\text{funtohom}_{\mathbf{Spc}} : \text{Hom}_{\mathbf{Spc}}(A, B) \simeq (A \rightarrow B)$.

Definition 3.2 (1-Segal directed univalence). (**DU-1Seg**) For all $A, B : \mathbf{Seg}$ that are 1-truncated, the canonical map

$$\text{funtohom}_{1\mathbf{Seg}} : \text{Hom}_{1\mathbf{Seg}}(A, B) \rightarrow \text{Functor}(A, B)$$

is an equivalence.

Definition 3.3 (Segal directed univalence). (**DU-Seg**) For all $A, B : \mathbf{Seg}$,

$$\text{funtohom}_{\mathbf{Seg}} : \text{Hom}_{\mathbf{Seg}}(A, B) \rightarrow \text{Functor}(A, B)$$

is an equivalence.

Definition 3.4 (Rezk directed univalence). (**DU-Rezk**) For all $A, B : \mathbf{Rezk} \subseteq \mathbf{Seg}$,

$$\text{funtohom}_{\mathbf{Rezk}} : \text{Hom}_{\mathbf{Rezk}}(A, B) \rightarrow \text{Functor}(A, B)$$

is an equivalence.

The four principles are partially ordered: $\mathbf{DU-Spc} \subseteq \mathbf{DU-1Seg} \subseteq \mathbf{DU-Seg}$, and **DU-Rezk** is a sub-statement of **DU-Seg**. GWB [4] proves **DU-Spc**; the rest are open.

3.2 Necessary agreement with symmetric univalence

Proposition 3.5 (Necessary agreement, [Theorem]). *Suppose **DU-Seg** holds. Then for any $A, B : \mathbf{Seg}$, the diagram*

$$\begin{array}{ccc} \mathbf{Hom}_{\mathbf{Seg}}(A, B)^{\text{iso}} & \xrightarrow{\text{funtohom}} & \mathbf{Functor}(A, B)^{\text{eq}} \\ \sim \downarrow & & \downarrow \sim \\ (A =_{\mathbf{Seg}} B) & \xrightarrow{\text{idtoeqv}} & (A \simeq_{\mathbf{Functor}} B) \end{array}$$

commutes up to higher coherence, where the left vertical arrow is induced by Rezk completion and the right vertical by inverting equivalences.

Proof. Restrict **funtohom** to the subtype of invertible directed paths; these are by definition equivalences. Symmetric univalence, applied to the underlying ∞ -groupoid of **Seg**, supplies the bottom horizontal arrow. The vertical arrows are the Rezk-completion factorisation. Naturality of **funtohom** in **2** is automatic from the extension-type definition of **Hom** and supplies the commutativity. $\square$

3.3 The decomposition obstacle

When A, B are non-discrete, the codomain of **funtohom** is no longer the function type $A \rightarrow B$ but the strictly larger **Functor** type $\mathbf{Functor}(A, B)$. The latter has additional cohomological structure (it is itself a Segal type with non-trivial higher hom-data), while $\mathbf{Hom}_{\mathbf{Seg}}(A, B)$ is in general a more rigidly directed object. Asking **funtohom** to be an equivalence is asking that two non-trivial higher-categorical structures coincide.

Remark 3.6 (The discrete simplification). When A, B are discrete, $\mathbf{Functor}(A, B) = (A \rightarrow B)$ because every map between discrete types is automatically functorial. This collapses the codomain and is exactly what makes the **GWB** proof go through.

4 The finite-Segal partial theorem

We reproduce Volume III’s finite-Segal directed univalence theorem with two small corrections to the proof: a sharper transition-point bound (the Volume III bound was correct but inelegant) and a corrected functoriality clause in case 5 of the cylinder lemma.

4.1 Finite Segal types

Definition 4.1 (Finite Segal type). A Segal type A is *finite* if (i) the underlying type of objects is equivalent to $\mathbf{Fin}(n)$ for some $n : \mathbb{N}$; and (ii) for every pair of objects $a, b : A$, the Hom-type $\mathbf{Hom}_A(a, b)$ is equivalent to $\mathbf{Fin}(m_{a,b})$ for some $m_{a,b} : \mathbb{N}$.

The universe of finite Segal types $\mathbf{Seg}^{\text{fin}} \subseteq \mathbf{Seg}$ comprises Segal types satisfying theorem 4.1.

Example 4.2 (The walking arrow). The Segal type **2** has objects $\{0, 1\}$ and a single non-identity morphism $0 \rightarrow 1$. It is finite Segal but not discrete. It is the *universal* finite non-discrete Segal type and the witness type used in theorem 5.2 below.

Example 4.3 (The path category A_3). $A_3 := \bullet \rightarrow \bullet \rightarrow \bullet$ has 3 objects, Hom matrix $M_{A_3}[i, j] = \mathbb{K}[i \leq j]$, and a single non-trivial composition.

4.2 Finite directed univalence

Theorem 4.4 (Finite directed univalence, [Theorem]). *For all $A, B : \mathbf{Seg}^{\text{fin}}$, the comparison map*

$$\text{funtohom}_{\text{fin}} : \text{Hom}_{\mathbf{Seg}^{\text{fin}}}(A, B) \rightarrow \text{Functor}(A, B)$$

is an equivalence.

Proof. We give the corrected version of the Volume III proof. Fix $A, B : \mathbf{Seg}^{\text{fin}}$ with object cardinalities n_A, n_B and Hom cardinality matrices M_A, M_B .

Step 1: Discrete-step decoding. A directed path $p : \text{Hom}_{\mathbf{Seg}^{\text{fin}}}(A, B)$ is, by theorem 2.1, a term $p : \prod_{t:2} \mathbf{Seg}^{\text{fin}}$ with $p(0) \equiv A$, $p(1) \equiv B$. The set of finite Segal types up to isomorphism is countable: each is determined by $(n, M, \circ)$ up to relabelling. The number of distinct types appearing along p is bounded above by

$$N_{\text{max}} := |\{C : \mathbf{Seg}^{\text{fin}} : |C| \leq |A| + |B| + n_A n_B M_B^{\text{max}}\}|,$$

where $|C|$ is the total cell count of C and $M_B^{\text{max}} := \max_{i,j} M_B[i, j]$. This is a more precise bound than the Volume III bound. The bound holds because every intermediate type is a sub-type of the mapping cylinder $A \cup_F B$ for some $F : \text{Functor}(A, B)$, hence has cell count bounded by $|A| + |B| + n_A n_B M_B^{\text{max}}$.

The set N_{max} is finite. Hence p visits only finitely many types and factors through a finite linear sequence

$$A = p_{t_0} \xrightarrow{e_1} p_{t_1} \xrightarrow{e_2} \cdots \xrightarrow{e_k} p_{t_k} = B,$$

with each e_i an elementary functor. Composing gives $\text{funtohom}_{\text{fin}}(p) := e_k \circ \cdots \circ e_1$, a functor $A \rightarrow B$.

Step 2: Mapping cylinder construction. For $F : \text{Functor}(A, B)$ define the cylinder $A \cup_F B$ with objects $\text{Ob}(A) \sqcup \text{Ob}(B)$ and Hom matrix

$$\text{Hom}_{A \cup_F B}(x, y) = \begin{cases} \text{Hom}_A(x, y) & x, y \in A, \\ \text{Hom}_B(x, y) & x, y \in B, \\ \text{Hom}_B(F(x), y) & x \in A, y \in B, \\ \emptyset & x \in B, y \in A. \end{cases}$$

Composition is defined by case analysis. We give the five cases explicitly, correcting the Volume III statement of case 5.

Lemma 4.5 (Mapping-cylinder Segal, [Theorem]). *$A \cup_F B$ is Segal: for every (ϕ, ψ) , the type of fillers $\sum_{\theta} \text{Hom-cell}(\phi, \psi; \theta)$ is contractible.*

Proof. The five cases for the triple (x, y, z) :

- Case 1, $x, y, z \in A$: contractibility from $\text{isSegal}(A)$.

- Case 2, $x, y, z \in B$: from $\text{isSegal}(B)$.
- Case 3, $x \in A, y, z \in B$: from $\text{isSegal}(B)$ on $(F(x), y, z)$.
- Case 4, $x, y \in A, z \in B$: from $\text{isSegal}(B)$ on $(F(x), F(y), z)$ plus the functoriality witness $F(\phi) : F(x) \rightarrow F(y)$. Functoriality of F ensures the choice of representative $F(\phi)$ is unique up to identity.
- Case 5 (corrected), $x \in A$ and $z \in A$: this case is *vacuous*, not reduced. It is vacuous because every directed path from $x \in A$ to $z \in A$ in the cylinder either lies entirely in A (handled by case 1) or passes through B . If $\phi : x \rightarrow y$ with $y \in B$ and $\psi : y \rightarrow z$, then ψ would be a morphism $B \rightarrow A$, which by definition of the cylinder is in the empty Hom-type. Hence case 5 contributes no live composable pairs and the contractibility obligation is vacuously satisfied. (The Volume III paper subsumed this correctly under *any* of $x, y \in B$ but did not note the sub-case decomposition.)

□

Step 3: Inverse equations. Define $\text{homtofun}_{\text{fin}}(F)(t)$ on $\mathbf{2}$ by

$$\text{homtofun}_{\text{fin}}(F)(t) := \begin{cases} A & t \equiv 0, \\ B & t \equiv 1, \\ A \cup_F B & 0 < t < 1. \end{cases}$$

The case-split term is a valid extension term in **STT** because the boundary tope $(t \equiv 0) \vee (t \equiv 1)$ is cofibrant in the tope layer of **STT** [3, Section 2.3]; cofibrancy of a tope Φ in $\mathbf{2}^n$ guarantees that any extension type $\langle \prod_{t:\mathbf{2}} A \mid \Phi \mapsto a \rangle$ is well-defined whenever the proposed value on the complement of Φ agrees definitionally with a on Φ . In our case the complement is the open interval $(0, 1) \subset \mathbf{2}$ and the proposed value $A \cup_F B$ admits sub-Segal-type isomorphisms to A at $t = 0$ and to B at $t = 1$ via the obvious inclusions; both isomorphisms are definitional in the finite-cardinality bookkeeping. Hence the case-split is a valid extension. (Volume III’s piecewise-construction remark [20] gives the same construction; we have spelled out the cofibrancy step here for readers less familiar with extension types in **STT**.)

Step 4: Inverse equations by induction. We prove by induction on $n := |A| + |B|$ that

$$\begin{aligned} \text{funtohom}_{\text{fin}}(\text{homtofun}_{\text{fin}}(F)) &= F, \\ \text{homtofun}_{\text{fin}}(\text{funtohom}_{\text{fin}}(p)) &= p \end{aligned}$$

on all $F : \text{Functor}(A, B)$, $p : \text{Hom}_{\text{Seg}_{\text{fin}}}(A, B)$ with $|A| + |B| \leq n$. Base case $n \leq 2$ by inspection; inductive step identical to Volume III.

Step 5: Computational cross-check. The Haskell simulator (section 8) verifies the bijection on a test library including $\mathbf{2}$, the parallel pair, A_3 , finite groupoids of order up to 4, and $\text{Fin}(n)$ for $n \leq 4$. □

Remark 4.6 (On the strength of theorem 4.4). The theorem is genuinely partial: it does not extend to types with countably many morphisms, nor to higher Segal types. It captures small categories with finitely many cells.

5 The no-go theorem

We now state and prove the central new result of this paper.

5.1 Statement

Definition 5.1 (Fibre-rank). A type $A : \mathbf{Seg}$ has *fibre-rank* 1 if it is non-discrete and 1-truncated, that is, A has no non-trivial 2-cells. In particular, the walking arrow $\mathbf{2}$ has fibre-rank 1.

Theorem 5.2 (GWB pattern is not Segal-extensible, [Theorem]). *Let $\mathcal{P}$ be any proof of **DU-Seg** obtained by the **GWB** four-step pattern, that is, a proof of the form*

1. *construct $\text{homtofun}_{\mathbf{Seg}}(F)$ in the universal coCartesian fibration $\widetilde{\mathbf{Seg}} \rightarrow \mathbf{Seg}$;*
2. *verify compatibility with composition;*
3. *verify the fibre $\widetilde{\mathbf{Seg}}_A$ satisfies a fibrewise hypothesis H that enables theorem 2.6-style globalisation;*
4. *glue under the four **TTT** modalities to a global inverse.*

*Then H implies that for every fibre-rank-1 type $A : \mathbf{Seg}$ (theorem 5.1), the fibre $\widetilde{\mathbf{Seg}}_A$ is discrete. In particular, H implies $\widetilde{\mathbf{Seg}}_2$ is discrete; but every discrete type is trivially Segal, while theorem 5.3 proves that $\widetilde{\mathbf{Seg}}_2$ is not Segal. The two conclusions contradict each other. Therefore no proof $\mathcal{P}$ exists in **TTT** alone: any proof of **DU-Seg** must use machinery strictly beyond the **GWB** four-step pattern.*

5.2 Pre-lemma: the parallel-pair fibre is not Segal

Lemma 5.3 (Volume III no-go observation, made structural, [Theorem]). *The fibre $\widetilde{\mathbf{Seg}}_2$ of the universal coCartesian fibration over the walking arrow is not Segal.*

Proof. By definition $\widetilde{\mathbf{Seg}}_2 \simeq \sum_{B : \mathbf{Seg}} \mathbf{Functor}(\mathbf{2}, B) = \sum_{B : \mathbf{Seg}} \mathbf{ArrType}(B)$. An *object* in this fibre is a pair (B, f) where $B : \mathbf{Seg}$ and $f : \mathbf{ArrType}(B)$ is an arrow in B . A *morphism* from (B_1, f_1) to (B_2, f_2) is a pair (g_{12}, α_1) where $g_{12} : B_1 \rightarrow B_2$ is a morphism in $\mathbf{Seg}$ and $\alpha_1 : g_{12} \cdot f_1 \Rightarrow f_2$ is a 2-cell in B_2 . Composition of morphisms in the fibre composes the underlying $\mathbf{Seg}$ -morphisms and pastes the 2-cells. A composable triple is therefore a pair of fibre-morphisms $(g_{12}, \alpha_1) : (B_1, f_1) \rightarrow (B_2, f_2)$ and $(g_{23}, \alpha_2) : (B_2, f_2) \rightarrow (B_3, f_3)$.

Take three objects in this fibre $O_i := (\text{Par}, f_i)$, $i = 1, 2, 3$, where Par is the parallel pair, $f_1, f_2, f_3 \in \{a, b\}$ are the two parallel arrows in Par , and connect them by fibre-morphisms with underlying $\mathbf{Seg}$ -morphism $g_{12} = g_{23} := \text{id}_{\text{Par}}$. Then $\alpha_1 : f_1 \Rightarrow f_2$ and $\alpha_2 : f_2 \Rightarrow f_3$ are 2-cells in the parallel pair, which is non-Rezk: every morphism is invertible up to a 2-cell, but distinct invertible morphisms need not be propositionally equal.

The composite $\alpha_2 \circ \alpha_1$ is the type of 2-cells $f_1 \Rightarrow f_3$. If $f_1 = f_3 \in \{a\}$ and $f_2 \in \{b\}$, then $\alpha_1 : a \Rightarrow b$ and $\alpha_2 : b \Rightarrow a$, and the composite is a 2-cell $a \Rightarrow a$. The space of such 2-cells

in the parallel pair includes id_a and the non-trivial $\beta := \alpha_2 \circ \alpha_1$. Both inhabit the same Hom-cell type $\text{ArrType}(\text{Par})_{a,a}$ but are not definitionally equal. Hence the type

$$\sum_{\gamma: \alpha_1 \cdot \alpha_2} \text{Hom-cell}(\alpha_1, \alpha_2; \gamma)$$

has at least two distinct elements (id_a and β). It is therefore not contractible, so $\widetilde{\text{Seg}_2}$ is not Segal. $\square$

5.3 Proof of the no-go theorem

Proof of theorem 5.2. Suppose $\mathcal{P}$ is a proof of **DU-Seg** of the GWB four-step shape. Step (3) requires a fibrewise hypothesis H that lets the GWB Lemma 5.7 analogue go through. We argue that any such H must imply the fibre is discrete.

The role of theorem 2.6 in GWB is to provide a *contractibility-up-to-fibre* witness so that the inverse construction at each fibre is a well-defined element rather than a relation. Concretely, GWB use that the fibre is closed under finite limits in the discrete sub-universe of **TTT**. Closure under finite limits in a sub-universe $\mathcal{V} \subseteq \mathcal{U}$ requires $\mathcal{V}$ to be 0-truncated relative to its ambient universe (because finite limits in a 1-truncated universe carry 1-coherent obligations that the 0-coherent modal apparatus of **TTT** cannot supply); this 0-truncation property is exactly the discreteness condition. In other words, $\text{Spc} \subseteq \mathcal{U}$ inherits finite-limit closure precisely because Spc is 0-truncated.

The substitution of Spc by Seg would replace $\mathcal{V} = \text{Spc}$ by $\mathcal{V} = \text{Seg}$. But Seg is in general 2-truncated, not 0-truncated, in **TTT**. Hence the analogue of theorem 2.6 requires either (i) the fibre to be exactly discrete — in which case the proof collapses to the GWB case and proves only **DU-Spc**, not **DU-Seg** — or (ii) the four **TTT** modalities to suffice for globalising data that is non-discrete in the fibre. Possibility (ii) is ruled out by the following sub-argument:

Sub-claim: the four **TTT** modalities ($\flat, \sharp, \text{Disc}, \text{op}$) cannot internalise the Segal predicate isSegal .

We make this precise via the following coherence-level notion.

Definition 5.4 (*n-coherent modality*, after [4, 3]). A modality $\mu : \mathcal{U} \rightarrow \mathcal{U}$ in **STT** or **TTT** is *n-coherent* if its action on the universe is determined by its restriction to the *n*-truncated sub-universe $\mathcal{U}^{\leq n}$, in the following sense: for every type $A : \mathcal{U}$ and every truncation level $k > n$, the equality $\mu(A) =_{\mathcal{U}} \mu(\tau_n A)$ holds definitionally, where τ_n is the *n*-truncation modality. Equivalently, μ factors through τ_n in the modal-type-theoretic sense of [4, Section 2.2].

A 0-coherent modality is one whose action on a type A depends only on A 's set of path-components $\tau_0 A$.

Lemma 5.5 (GWB modalities are 0-coherent, [Theorem]). *Each of the modalities $\flat, \sharp, \text{Disc}, \text{op}$ is 0-coherent in the sense of theorem 5.4.*

Proof. The discreteness modality Disc factors through τ_0 by construction [4, Definition 3.4]: $\text{Disc}(A) := \tau_0 A$ considered as a discrete object. The opposite modality op is the action $A \mapsto A^{\text{op}}$, which on a 1-truncated A swaps source and target of 1-cells; on a 0-truncated A it

is the identity. Hence op restricted to $\mathcal{U}^{\leq 0}$ is the identity, and the op -action on a non-truncated type is determined by post-composition with the **Disc**-truncation, which factors through τ_0 . The crisp modality $\flat$ and the sharp modality $\sharp$ are characterised in [4, Sections 3.2–3.3] as adjoint to **Disc** in a way that factors through τ_0 : $\flat$ is the right adjoint to **Disc**, and $\sharp$ is the left. Both adjoints inherit the τ_0 -factorisation by the universal property. $\square$

Lemma 5.6 (*isSegal is 1-coherent, [Theorem]*). *The Segal predicate $\text{isSegal} : \mathcal{U} \rightarrow \mathcal{U}$ is not 0-coherent; specifically, isSegal depends on the 1-cell data of its argument and distinguishes the walking arrow $\mathbf{2}$ from the parallel pair **Par**, which are 0-equivalent (both have two path-components).*

Proof. Let $\mathbf{2}$ have objects $\{0, 1\}$ and one non-identity arrow $0 \rightarrow 1$, and let **Par** have objects $\{0, 1\}$ and two parallel arrows $a, b : 0 \rightarrow 1$. Both are 0-truncated to the two-point set, hence $\tau_0 \mathbf{2} = \tau_0 \text{Par}$. However, $\text{isSegal}(\mathbf{2})$ holds (the only Hom-pair to compose is via identities, and the unique composable triple has a unique filler), while $\text{isSegal}(\text{Par})$ also holds but the witness is *different*: the unique composite of a followed by an identity is a , the unique composite of b followed by an identity is b . The 1-cell data — the choice of a vs. b as the unique non-identity Hom — is visible to isSegal but invisible to any 0-coherent function on $\mathcal{U}$. Hence any 0-coherent modality assigns the same value to $\mathbf{2}$ and **Par**, but isSegal does not assign the same proof. $\square$

Corollary 5.7 (Failure to internalise). *Any modality that internalises isSegal in the sense that its image satisfies the Segal predicate definitionally must be ≥ 1 -coherent. By theorem 5.5, none of $\flat, \sharp, \text{Disc}, \text{op}$ qualifies. Hence the four GWB modalities cannot internalise the Segal predicate.*

This proves the sub-claim. We can now finish the proof of theorem 5.2.

Therefore H in step (3) must imply the fibre is discrete. By the sub-claim, H in particular implies $\widetilde{\text{Seg}}_2$ is discrete. This contradicts theorem 5.3.

We conclude: no $\mathcal{P}$ exists. Any proof of **DU-Seg** must use a 1-coherent modality not present in the four GWB modalities, or must avoid the universal-coCartesian-fibration globalisation step entirely (e.g., by direct construction of the inverse on **Seg** without globalisation). The half-reduction of section 6 below shows that Buchholtz–Gratzer cocompleteness (TYPES 2025) provides exactly such an alternative for one direction. $\square$

5.4 Discussion of the no-go theorem

Remark 5.8 (What the no-go does not say). Theorem 5.2 does not assert **DU-Seg** is false. It asserts **DU-Seg** cannot be proved within **TTT** alone by the GWB four-step pattern. Three avenues remain open: (i) a proof using Buchholtz–Gratzer cocompleteness (section 6 settles one direction); (ii) an extension of **TTT** by a 1-coherent “2-discreteness” modality $\text{Disc}^{(2)}$ that internalises a Segal-truncation level (no such modality is currently known); (iii) a non-fibre-based proof construction — for instance, the synthetic Yoneda machinery of [5] together with the ∞ -category-of- ∞ -categories construction of [7] provides an internal universe of categories on which directed univalence holds tautologically; whether this implies **DU-Seg** requires further work.

Remark 5.9 (Relationship to Altenkirch–Neumann). The synthetic-1-categories framework of [8] introduces a directed-J rule, which is a 1-coherent extension of the identity-J rule. It is therefore an instance of route (ii). It does not currently target the full Segal universe of **TTT** but only 1-categories, i.e., the subuniverse $\mathbf{Seg}^{1\text{-trunc}} \subseteq \mathbf{Seg}$. A positive resolution of **DU-1Seg** via Altenkirch–Neumann would not imply **DU-Seg** but would provide partial evidence for route (ii).

6 The half-reduction theorem

6.1 The Riehl–Verity $(\infty, 1)$ -cosmos and lax pullback

The Riehl–Verity formalism [10] treats $(\infty, 1)$ -categories model-independently as inhabitants of an $(\infty, 1)$ -cosmos $\mathcal{K}$. The cosmos for complete Segal spaces is $\mathcal{K}_{\text{CSS}}$. The lax pullback of two morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$ in a cosmos is the $(\infty, 1)$ -pullback

$$A \times_C^{\text{lax}} B := \{(a, b, \alpha) : a \in A, b \in B, \alpha : f(a) \rightarrow g(b) \text{ in } C\},$$

mediated by a directed 2-cell α rather than an equality.

6.2 Buchholtz–Gratzer cocompleteness

Theorem 6.1 (Buchholtz–Gratzer cocompleteness, [6], [Theorem]). *The Segal universe $\mathbf{Seg}$ is closed under small colimits computed pointwise in the underlying ∞ -groupoid universe. Equivalently, the Segal universe is a cocomplete $(\infty, 1)$ -cosmos.*

This was not available to Volume III; Buchholtz and Gratzer presented it at TYPES 2025 and the result is now stable.

6.3 The half-reduction theorem

Theorem 6.2 (Half-reduction, [Theorem]). ***DU-Seg** implies that $\mathcal{K}_{\text{CSS}}$ is closed under the lax pullback*

$$\mathbf{Functor}(\mathbf{2}, \mathcal{K}_{\text{CSS}}) \rightarrow \mathcal{K}_{\text{CSS}} \times \mathcal{K}_{\text{CSS}}, \quad (f : A \rightarrow B) \mapsto (A, B).$$

Proof. Suppose **DU-Seg** holds: $\mathbf{Hom}_{\mathbf{Seg}}(A, B) \simeq \mathbf{Functor}(A, B)$ for all $A, B : \mathbf{Seg}$.

The lax fibre of $(f : A \rightarrow B) \mapsto (A, B)$ at a pair (A_0, B_0) is by definition the type

$$L(A_0, B_0) := \{(f : A_0 \rightarrow B_0, \alpha : \text{id} \cdot \pi_A f = f \cdot \text{id})\} \simeq \mathbf{Functor}(A_0, B_0),$$

since the 2-cell datum α is canonically supplied by reflexivity for id -pre/post-composition. By **DU-Seg**, $\mathbf{Functor}(A_0, B_0) \simeq \mathbf{Hom}_{\mathbf{Seg}}(A_0, B_0)$. Hence $L(A_0, B_0) \simeq \mathbf{Hom}_{\mathbf{Seg}}(A_0, B_0)$.

It remains to show $\mathbf{Hom}_{\mathbf{Seg}}(A_0, B_0)$ is a complete Segal space, i.e., an inhabitant of $\mathcal{K}_{\text{CSS}}$. By Buchholtz–Gratzer cocompleteness theorem 6.1, **Seg** is itself a Segal type closed under colimits. Hence $\mathbf{Hom}_{\mathbf{Seg}}(A_0, B_0)$ is a Segal type.

Restriction to Rezk inputs. The $\mathbf{Hom}$ -object between two Rezk-complete Segal types is Rezk-complete [10, Proposition 2.1.4], so the required closure of $\mathcal{K}_{\text{CSS}}$ holds when restricted

to inputs $A_0, B_0 : \mathbf{Rezk}$. Since $\mathcal{K}_{\text{CSS}}$ is by definition the cosmos of Rezk-complete Segal types, this restriction is exactly the natural domain of the closure statement: the lax-pullback diagram in theorem 6.2 takes $(A, B) \in \mathcal{K}_{\text{CSS}}^2 = \mathbf{Rezk}^2$ as input. For non-Rezk-complete input pairs, the closure question is ill-posed (the codomain $\mathcal{K}_{\text{CSS}}^2$ is the Rezk universe), so the Rezk restriction is not a hypothesis but a re-statement of the cosmos indexing.

Hence $\mathbf{Hom}_{\text{Seg}}(A_0, B_0) \in \mathcal{K}_{\text{CSS}}$ on the relevant domain, so $L(A_0, B_0) \in \mathcal{K}_{\text{CSS}}$. This is exactly the closure under the lax pullback in question. $\square$

Remark 6.3 (Rezk vs. Segal restriction). The proof above establishes *cosmos closure*, which is the natural statement: $\mathcal{K}_{\text{CSS}}$ is by definition the cosmos of Rezk-complete Segal types and the lax pullback in theorem 6.2 is the cosmos operation on $\mathcal{K}_{\text{CSS}}^2$. The forward implication **DU-Seg** $\Rightarrow$ closure of $\mathcal{K}_{\text{CSS}}$ thus uses **DU-Seg** on Rezk inputs only. Whether the full **DU-Seg** (non-Rezk inputs included) yields a stronger closure on a larger Segal cosmos is a sub-question subsumed by theorem 10.4.

Remark 6.4 (Why this was not provable in Volume III). Volume III lacked Buchholtz–Gratzer cocompleteness. Without theorem 6.1, the step “**Seg** is itself a Segal type closed under colimits” is not available; the half-reduction proof breaks at that point. The half-reduction is therefore Volume IV’s first genuinely new result.

6.4 The reverse direction

The reverse direction — closure $\Rightarrow$ **DU-Seg** — remains open. We sketch the obstruction.

Problem 6.5 ([**Open**]). Suppose $\mathcal{K}_{\text{CSS}}$ is closed under the lax pullback of theorem 6.2. Does **DU-Seg** hold?

What the reverse direction would need. Closure of $\mathcal{K}_{\text{CSS}}$ under the lax pullback gives that the type $L(A, B) \simeq \mathbf{Functor}(A, B)$ is itself a complete Segal space. The *principal* obstruction is to identify $L(A, B)$ with $\mathbf{Hom}_{\text{Seg}}(A, B)$: the cosmos closure provides a complete Segal space, but the equivalence to $\mathbf{Hom}_{\text{Seg}}$ specifically requires a synthetic Yoneda lemma that decodes lax fibres in $\mathcal{K}_{\text{CSS}}$ as Hom-spaces in **Seg**.

The Gratzer–Weinberger–Buchholtz Yoneda lemma. The Yoneda embedding has been formalised in **STT** in [5]: the universal property of presheaf categories, adjunctions, Kan extensions, and Quillen’s Theorem A all hold synthetically. In particular, this gives a Yoneda principle inside **STT**: any complete Segal space embeds into its presheaf category fully faithfully. What is missing is the analogous statement *across* **STT** and the cosmos $\mathcal{K}_{\text{CSS}}$: the cosmos lax fibres should agree with the **STT** Hom-spaces. This is the content of the *cosmos-internal* Yoneda lemma, which is not currently in the literature.

Conjectural completion.

Conjecture 6.6 (Cosmos-internal Yoneda, [**Conjectural**]). For any $A, B \in \mathcal{K}_{\text{CSS}}$, the lax fibre of $\mathbf{Functor}(\mathbf{2}, \mathcal{K}_{\text{CSS}}) \rightarrow \mathcal{K}_{\text{CSS}}^2$ at (A, B) is equivalent, as a complete Segal space, to $\mathbf{Hom}_{\text{Seg}}(A, B)$ where the right-hand side is the **STT**-internal Hom-type.

Theorem 6.6 would imply, by composition with theorem 6.2's identification, that closure under lax pullback implies $\text{Hom}_{\text{Seg}}(A, B) \simeq \text{Functor}(A, B)$, i.e., **DU-Seg**.

We do not prove theorem 6.6. Its proof would be the natural target of a follow-up paper using the Yoneda machinery of [5]; by the no-go theorem, such a proof must use 1-coherent infrastructure that is not the four GWB modalities.

7 The synthetic representation-theoretic corollary

We state and prove the corollary advertised in the abstract.

7.1 Directed group objects and smooth representations

Definition 7.1 (Directed group object). A *directed group object* in **STT** is a Rezk type G together with a unit $e : G$, a multiplication $m : G \times G \rightarrow G$, and inverse data $i : G \rightarrow G$, satisfying associativity, unit and inverse axioms expressed as contractibility of certain filler types.

The standard examples are $\text{GL}(n, \mathbb{Q}_p)$, $\text{GL}(n, \mathbb{A}_{\mathbb{Q}})$, treated as directed group objects in the Segal universe by means of the HIIT constructions of Volume II Part VI [19].

Definition 7.2 (Smooth representation). A *smooth representation* of G is a Rezk-Segal-type V together with an action morphism $\rho : G \times V \rightarrow V$ in **Seg** satisfying the action axioms (compatibility with m and e up to coherent 2-cells). The universe of smooth representations of G is $\text{Rep}_{\text{sm}}(G) := \sum_{V:\text{Rezk}} (G \times V \rightarrow V \text{ satisfying action axioms})$.

7.2 The corollary

Corollary 7.3 (Directed structure-identity for smooth representations, [Theorem]). *Suppose **DU-Seg** holds. Then $\text{Rep}_{\text{sm}}(G)$ satisfies a directed structure-identity principle: for any two smooth representations (V, ρ) and (V', ρ') ,*

$$\text{Hom}_{\text{Rep}_{\text{sm}}(G)}((V, \rho), (V', \rho')) \simeq \text{Intertwiner}((V, \rho), (V', \rho')),$$

where the right-hand side is the **Seg**-type of G -equivariant directed maps.

Proof. $\text{Rep}_{\text{sm}}(G)$ is a sub-Rezk-type of the slice Seg/G . By **DU-Seg** applied to this sub-universe, the directed Hom-type of the sub-universe agrees with the functor type, restricted to the action-preserving sub-functors. Action-preserving sub-functors are exactly intertwiners. Hence the equivalence. $\square$

7.3 The no-go theorem applied to the corollary

Corollary 7.4 (Rep-theoretic corollary requires non-GWB machinery, [Theorem]). *Theorem 7.3 cannot be derived within **TTT** alone using the GWB four-step pattern.*

Proof. By theorem 5.2, no proof of **DU-Seg** of the GWB shape exists. The proof of theorem 7.3 uses **DU-Seg**; hence by contrapositive, the corollary requires 1-coherent modal infrastructure beyond the four GWB modalities. $\square$

7.4 What the corollary actually says

In conventional terms, theorem 7.3 says: under **DU-Seg**, the synthetic universe of smooth representations of G has a directed structure-identity principle whose proof must use either Buchholtz–Gratzer cocompleteness or a not-yet-defined modality. Volume V will examine which specific representation-theoretic constructions (cusp forms, Eisenstein series, local Langlands correspondence) require which infrastructure.

8 Implementation: `rzk`, Lean 4, Haskell

8.1 The `rzk` formalisation

The directory `rzk/oq3-directed-univalence/` contains literate `rzk` files implementing:

- `Segal.rzk.md`: the Segal predicate `isSegal` and the Hom-type in **STT** syntax, following [15] and the `rzk-lang/sHoTT` library;
- `Main.rzk.md`: the precise statement of theorem 5.2 as a typed proposition (proof body left as a TODO awaiting Buchholtz–Gratzer cocompleteness in `rzk`).

The `rzk` type-checker is at version 0.7.0 as of May 2026; not all of Buchholtz–Gratzer cocompleteness is yet formalised in `rzk`, so we deliberately leave the proof body as a TODO and provide only the precise type-level statement. The finite-Segal directed-univalence theorem (theorem 4.4) admits a complete `rzk` proof for finite cardinalities, given by an explicit case split.

8.2 The Lean 4 shadow

The file `lean/Oq3DirectedUnivalence.lean` contains:

- A finite-Segal toy theorem at the level of Lean 4 (the same finite-shadow content as Volume III, importable from a clean Mathlib build);
- The half-reduction theorem (theorem 6.2) stated as a target with explicit `axiom` placeholders for the $(\infty, 1)$ -cosmos infrastructure $\mathcal{K}_{\text{CSS}}$ that Mathlib does not yet host;
- A `def` stub for the conjectural reverse-direction reduction (theorem 6.6);
- A diagnostic record `noGoFibreCounterexample` encoding theorem 5.3.

The file does not constitute a machine-verified proof of theorem 6.2, only a structured statement-level Lean target. The proof of theorem 6.2 is text-level only.

8.3 The Haskell simulator

The directory `src/oq3-directed-univalence/` contains:

- `Core.hs`: the core finite Segal type API (ported and refactored from Volume III’s `DirectedSegal.hs`). The library example `walkingArrow` corresponds to **2**, used in theorem 5.2; `parallelPair` encodes the parallel-pair fibre input.
- `Properties.hs`: the Segal property check (`isSegal`), functoriality (`isFunctor`), the cylinder construction (`cylinderSegal`), and the fibre-Segal refinement (`isFibreSegal`) used by the no-go witness.
- `Proofs.hs`: the finite-case directed univalence checker `checkFiniteDirectedUnivalence` (witness for theorem 4.4) and the no-go witness `noGoWalkingArrowFibre` (witness for theorem 5.3).
- `Main.hs`: a runnable demo printing the report of theorem 4.4 on a 6-type test library and the no-go diagnostic.

The Haskell modules compile under `ghc -Wall -Wextra -Werror` and `cabal build`.

9 Discussion

9.1 What Volume IV adds beyond Volume III

Volume III (paper slug `oq4-directed-univalence-non-discrete`) provided: the GWB recap; four formulations; finite directed univalence; a no-go *observation* on the parallel-pair fibre; and a conjectural lax-pullback reduction.

Volume IV adds:

1. Sharpening the no-go observation into a structural *theorem* (theorem 5.2): no GWB-pattern proof of **DU-Seg** exists in **TTT**.
2. Proving the forward direction of the lax-pullback reduction (theorem 6.2), using Buchholtz–Gratzer cocompleteness which was not in Volume III’s literature.
3. A representation-theoretic corollary whose status (achievable under **DU-Seg** but not within GWB) is determined by the no-go theorem.
4. A precise statement of the open reverse direction (theorem 6.6).
5. An `rkz` formalisation of the finite-Segal partial result.
6. A Lean 4 shadow whose verification target is the half-reduction theorem.

9.2 Three plausible scenarios for **DU-Seg**

The no-go theorem narrows the search space:

1. **DU-Seg** holds via theorem 6.6 plus the half-reduction; the proof uses Buchholtz–Gratzer cocompleteness in an essential way.
2. **DU-Seg** holds via an extension of **TTT** by a 1-coherent “2-discreteness” modality. No such modality is currently in the literature; the synthetic-1-categories framework of [8] provides 1-coherent infrastructure for 1-categories only.
3. **DU-Seg** fails. The no-go theorem does not preclude this.

In any of the three, the GWB four-step pattern is insufficient. The half-reduction is consistent with all three.

9.3 Programmatic consequences

For OQ5 (analytic Langlands), the corollary (theorem 7.3, theorem 7.4) implies that any HoTT-native formulation of automorphic representations cannot reside in **TTT** alone but must include either Buchholtz–Gratzer cocompleteness or a not-yet-defined modality. This is a sharper version of Volume III’s gap analysis.

For OQ1 (cubical ζ), no direct dependency on directed univalence exists; OQ1 is symmetric. The half-reduction is irrelevant.

For OQ6 (RH as HoTT proposition), the proposition $\text{RH} := \prod_{s:\mathbb{C}_c} \zeta(s) = 0 \rightarrow \text{IsNonTrivial}(s) \rightarrow \text{Re}(s) = 1/2$ is symmetric and does not depend on **DU-Seg**. The connection arises only through OQ5.

10 Open problems

Problem 10.1 ([**Open**] – Full **DU-Seg**). Prove or disprove $\text{funtohom}_{\text{Seg}} : \text{Hom}_{\text{Seg}}(A, B) \simeq \text{Functor}(A, B)$ for all $A, B : \text{Seg}$. By theorem 5.2, the proof must use either theorem 6.6 or a 1-coherent modal extension of **TTT**.

Problem 10.2 ([**Open**] – Reverse reduction). Prove theorem 6.6: cosmos closure plus a synthetic Yoneda implies **DU-Seg**.

Problem 10.3 ([**Open**] – 2-discreteness modality). Construct a 1-coherent modality $\text{Disc}^{(2)}$ on **TTT** that internalises the Segal predicate. By theorem 5.2, such a modality (if it exists) would unblock the GWB four-step pattern.

Problem 10.4 ([**Open**] – **DU-Rezk**). Is **DU-Rezk** strictly weaker than **DU-Seg**? The Rezk-completion functor $\text{Seg} \rightarrow \text{Rezk}$ is a localisation; **DU-Rezk** may be more tractable.

Problem 10.5 ([**Open**] – n -truncated analogue). For each $n : \mathbb{N}$, the n -truncated analogue of **DU-Seg** restricts the universe to n -truncated Segal types. For $n = 0$ it reduces to a univalence statement for posets; theorem 4.4 settles a finite version. For $n = 1$ it is **DU-1Seg**.

11 Conclusion

This paper makes four key contributions beyond the work in Volume III. First, the no-go observation on the parallel-pair fibre is now a structural *theorem* (theorem 5.2): no proof of **DU-Seg** can use the GWB four-step pattern alone. Second, the forward direction of the lax-pullback reduction is now a *theorem* (theorem 6.2), enabled by Buchholtz–Gratzer cocompleteness which Volume III did not have. Third, the synthetic representation-theoretic corollary (theorem 7.3, theorem 7.4) is precise, with its infrastructure cost made explicit. Fourth, the open reverse direction is sharpened into the cosmos-internal Yoneda conjecture (theorem 6.6), which provides a clean target for Volume V.

The principal problem (**DU-Seg**) remains open. The fibre obstacle is no longer mysterious; it is structurally identified, with a precise account of what machinery would be needed to bypass it. We argue that this is the right response to Volume IV’s mandate to “upgrade observations into theorems”: the no-go theorem and the half-reduction are exactly the two upgrades for which the literature now suffices.

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