Identifying the $(\infty, 1)$-Topos for Analytic Langlands: A Rasekh-Axiom Audit of $\mathcal{E}_{\mathrm{cond}}$ with Cardinal-Stratification Caveats

Matthew Long
The YonedaAI Collaboration
YonedaAI Research Collective
Chicago, IL
research@yonedaai.com · https://yonedaai.com

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Abstract

We revisit the central topos-theoretic question of the HoTT-RH programme, raised as Open Question 5 of Volume III: in which elementary $(\infty, 1)$-topos do automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ live natively? Volume III sketched the candidate $\mathcal{E}_{\mathrm{cond}}$, the $(\infty, 1)$-topos of light condensed anima of Clausen and Scholze, and asserted its elementarity by appeal to the Grothendieck-implies-elementary slogan. The slogan is essentially correct: by Lurie HTT and Shulman's theorem on strict univalent universes [15], $\mathcal{E}_{\mathrm{cond}}$ admits a sequence of internal univalent universes $\mathcal{U}^{\kappa}$ classifying κ-small fibrations for every regular cardinal κ above an inaccessible threshold. This is enough to verify Rasekh's elementarity axioms (E1)–(E5) [6] at each fixed universe level. Our principal positive result is therefore: $\mathcal{E}_{\mathrm{cond}}$ *is an elementary* $(\infty, 1)$*-topos in the universe-stratified Rasekh sense* (Theorem 3.6).

Volume III's actual gap is more subtle and concerns the choice of κ. The natural automorphic-representation objects $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, $\mathbb{A}_{\mathbb{Q}}$, and the smooth-representation categories sit at different cardinal levels of the universe stratification, and Volume III did not determine which κ contains them all simultaneously. We make this gap explicit (Theorem 4.2, Theorem 4.4) and identify a precise *universe-coherence question* (UCQ) which, if answered positively, would let one work in a single fixed κ. We do not answer UCQ in this paper; resolving it is the principal Volume V deliverable.

We then study a cohesive supplementary structure on $\mathcal{E}_{\mathrm{cond}}$ obtained by adjoining the standard shape-flat-sharp modality triple in the spirit of Schreiber-Shulman cohesive HoTT [17, 16]. The flat modality $\flat$ is the constant-sheaf functor; its image is An, the discrete sub-universe. Crucially, the cohesive structure does *not* collapse $\mathcal{E}_{\mathrm{cond}}$: the analytic objects $\mathbb{A}_{\mathbb{Q}}$, $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ are non-discrete (not $\flat$-modal) and sit in the ambient cohesive topos with their full topological data preserved. We give a HoTT-internal definition of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$-automorphic representations directly in $\mathcal{E}_{\mathrm{cond}}$ (Theorem 6.4),

using the condensed-category-theoretic notion of profinite sub-condensed-group (Theorem 6.2) for the smoothness condition; show that the local liquid/solid obstruction of Volume III generalises strictly (Theorem 7.1); and prove the framework composes with the directed-univalence machinery of OQ3 in the standard way (Theorem 8.2).

We provide a Lean 4 / Mathlib formalisation `Oq5LanglandsTopos.lean` stating the topos-axiom-check theorems with placeholder proofs, and Haskell modules under `src/oq5-langlands-topos/` demonstrating finite-cardinality toy structures (light-profinite tower, cohesive sheaf check, GL_2 class-sum Hecke commutativity) to provide concrete handles on the framework.

Per the Volume IV deliverable matrix, the verdict on OQ5 is: **(c) partial result with clear path to Volume V**. The honest claim is that $\mathcal{E}_{\mathrm{cond}}$ *is* elementary in the universe-stratified sense (positive direction); the cardinal level required for the automorphic-representation theory is open (UCQ); and the local obstruction generalises (negative direction).

Contents

Pi-disc

1 Introduction

Pi-disc

1.1 The Volume III question, sharpened

Pi-disc
Pi-disc

Volume III of the HoTT-Riemann Hypothesis programme [2] formulated six open questions toward a HoTT-native proof of the Riemann Hypothesis. OQ5 asks: *which elementary* $(\infty, 1)$*-topos in the sense of Rasekh [6] most naturally hosts automorphic representations of* $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$*?* Volume III's answer was a candidate, $\mathcal{E}_{\mathrm{cond}} := \mathrm{LCond}_{(\infty,1)}$, the $(\infty, 1)$-topos of light condensed anima of Clausen and Scholze [10, 11]. The Volume III paper labelled this candidate's elementarity [**Provable**] and gave a one-line proof: every Grothendieck $(\infty, 1)$-topos is elementary by Lurie HTT plus Volume II Part IV.

That one-line proof is, on more careful examination, essentially correct. By Shulman's theorem [15] ("All $(\infty, 1)$-toposes have strict univalent universes"), every Grothendieck $(\infty, 1)$-topos admits, for every Grothendieck universe level κ above an inaccessible threshold, an internal κ-univalent universe object $\mathcal{U}^\kappa$ classifying κ-small fibrations, satisfying univalence in the sense that its identity types compute equivalences. This is enough to verify Rasekh's axioms (E1)–(E5) at each fixed universe level κ. We make this verification explicit in Section 3. In particular, $\mathcal{E}_{\mathrm{cond}}$ is an elementary $(\infty, 1)$-topos in the standard, universe-stratified Rasekh sense.

Volume III's claim was not wrong. What Volume III did not address—and what we sharpen here—is the question of *which* cardinal κ houses the natural objects of analytic Langlands ($\mathbb{A}_{\mathbb{Q}}$, $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, the categories of smooth and automorphic representations) so that one can work in a single fixed universe level. We call this the *universe-coherence question* (UCQ). We do not answer UCQ in this paper; we make it precise (Theorem 4.3) and show that Volume V will need to address it before HoTT-internalisation of analytic Langlands can be carried out at a single universe level.

1.2 What this paper claims

Pi-disc

Positive direction: stratified elementarity (Theorem 3.6, [Provable]). $\mathcal{E}_{\mathrm{cond}}$ *is an elementary* $(\infty, 1)$*-topos in the universe-stratified Rasekh sense.* For each Grothendieck universe level κ above an inaccessible threshold, axioms (E1)–(E5) hold relative to κ, with $\mathcal{U}^\kappa$ classifying κ-small fibrations. This is essentially Shulman's theorem applied to the Grothendieck topos $\mathrm{LCond}_{(\infty,1)}$, with the elementarity dictionary of Rasekh–Shulman [7, 15].

Negative direction: the universe-coherence gap (Theorem 4.2, [Conjectural]). *The cardinal level required to house all natural analytic-Langlands objects in* $\mathcal{E}_{\mathrm{cond}}$ *is not fixed by current literature.* The adele ring $\mathbb{A}_{\mathbb{Q}}$ is a colimit of small objects and lives at $\kappa = \aleph_1$; the smooth-representation category requires a higher level (we conjecture $\kappa = \aleph_\omega$); and the existence of a single κ housing all of these—the universe-coherence question UCQ—is currently open.

Cohesive supplement (Section 5, [Provable]). *Adjoining a shape modality* Π_{disc} *to* $\mathcal{E}_{\mathrm{cond}}$ *in the Schreiber-Shulman style does not collapse the analytic content.* Following the cohesive HoTT framework of [16, 17], the shape modality Π_{disc} is left adjoint to a flat modality

$\flat$, which is in turn left adjoint to a sharp modality $\sharp$. The flat modality picks out the discrete part of the topos: $\flat\,\mathcal{E}_{\mathrm{cond}} \simeq \mathrm{An}$ as a full subcategory of constant sheaves. The analytic objects $\mathbb{A}_{\mathbb{Q}}$ and $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ are *not* in the image of $\flat$ (they are non-discrete condensed objects), but they sit in the ambient cohesive topos $\mathcal{E}_{\mathrm{cond}}$ with all their topological structure intact. The cohesive structure provides the modality triple as additional infrastructure on top of $\mathcal{E}_{\mathrm{cond}}$; it does not act on $\mathcal{E}_{\mathrm{cond}}$ destructively.

Composition with OQ3 (Theorem 8.2, [Provable]). *The cohesive structure on $\mathcal{E}_{\mathrm{cond}}$ admits the simplicial type theory (STT) directed interval and so composes with directed univalence in the sense of Gratzer–Weinberger–Buchholtz* [21, 22]. The shape modality commutes with the directed-interval-tensoring of STT.

Generalisation of the Volume III obstruction (Theorem 7.1, [Provable]). *The local liquid/solid obstruction of Volume III [3, Theorem 6.1] generalises to all ∞-localisations of $\mathcal{E}_{\mathrm{cond}}$ that preserve the analytic-ring structure.* The obstruction is not eliminated by passing to any cohesive variant.

1.3 Volume IV verdict on OQ5: partial result, path to Volume V

Pi-disc

Per the Volume IV deliverable matrix, OQ5 asks for one of three outcomes: (a) $\mathcal{E}_{\mathrm{cond}}$ IS an elementary $(\infty, 1)$-topos in the Rasekh sense, with proof and HoTT-internal automorphic representation definition; (b) $\mathcal{E}_{\mathrm{cond}}$ IS NOT, with counter-construction and an alternative; (c) partial result punted to Volume V. The honest verdict of this paper is *(c) partial result with clear path to Volume V*: positive direction (a) holds in the universe-stratified reading, and we give the HoTT-internal automorphic-representation definition (Theorem 6.4); but the universe-coherence question UCQ, which is the actual obstacle to a single-universe internalisation, is open and is the principal Volume V deliverable.

1.4 What this paper does not claim

Pi-disc

We do not prove the Riemann Hypothesis, the analytic Langlands correspondence, or the Clausen-Scholze functional-equation conjecture. We do not answer the universe-coherence question UCQ. We do not formalise the cohesive structure of $\mathcal{E}_{\mathrm{cond}}$ in Lean; the Lean file provides only the topos-axiom-check theorem statements as type-correct declarations with placeholder proofs. We do not address arithmetic Langlands at the level of L-function arithmetic.

1.5 Stratification policy

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Following the Volume III convention [2], every result is labelled:

Pi-disc

[Provable] complete proof, or proof modulo Volume II/III machinery and standard topos theory;

[Conditional] proof modulo a stated principle (e.g. UCQ) or external lemma;

[Conjectural] precise statement, no proof in this paper;

[Open] genuinely open, no claim either way.

1.6 Outline

Pi-disc

Section 2 sets the framework: HoTT, Rasekh elementarity (E1)–(E5), the universe-stratified reading, the Volume III input. Section 3 performs the Rasekh-axiom-by-Rasekh-axiom audit of $\mathcal{E}_{\mathrm{cond}}$ (positive direction). Section 4 introduces the universe-coherence question UCQ and discusses the cardinal-stratification gap. Section 5 adjoins a cohesive structure to $\mathcal{E}_{\mathrm{cond}}$ via the shape-flat-sharp modality triple, and explains why this does not collapse the analytic content. Section 6 gives the HoTT-internal definition of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$-automorphic representations directly in $\mathcal{E}_{\mathrm{cond}}$. Section 7 proves the local liquid/solid obstruction generalises. Section 8 composes with OQ3 directed univalence. Section 9 stratifies the main theorems. Section 10 describes the Lean formalisation. Section 11 describes the Haskell artifacts. Section 12 states what is not proved. Sections 13 and 14 discuss and conclude.

2 Mathematical Framework

Pi-disc

2.1 HoTT and the elementary $(\infty, 1)$-topos–HoTT correspondence

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We work in homotopy type theory (HoTT) as in [1], with univalent universes $\mathcal{U}_0 : \mathcal{U}_1 : \cdots$, identity types, Σ- and Π-types, higher inductive types, and the univalence axiom. Cubical Agda [26] provides an executable model. The Shulman correspondence [15] states that every Grothendieck $(\infty, 1)$-topos (in the sense of Lurie [14]) admits, for every regular cardinal κ above the inaccessibility threshold determined by the topos's site, a strictly univalent internal universe object $\mathcal{U}^\kappa$, and is therefore a model of HoTT (with universe stratification matching κ).

The Shulman result gives, for each fixed κ, an internal universe sufficient to interpret HoTT typing judgements involving κ-small types. The collection $\{\mathcal{U}^\kappa\}_\kappa$ of universes is itself external (it is indexed by an external ordinal, not by an internal type), but each universe is internal to the topos.

2.2 Rasekh's elementarity axioms

Pi-disc

Rasekh [6, 7] axiomatises elementary $(\infty, 1)$-toposes as follows. Fix a regular cardinal κ above the inaccessibility threshold of the relevant site. An $(\infty, 1)$-category $\mathcal{E}$ is an *elementary* $(\infty, 1)$*-topos at level* κ if:

(E1) Limits and colimits: $\mathcal{E}$ has all finite (homotopy) limits and finite (homotopy) colimits.

(E2) Local cartesian closure: for every morphism $f : Y \to X$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}_{/X} \to \mathcal{E}_{/Y}$ has a right adjoint $f_* = \Pi_f$.

(E3) Subobject classifier: $\mathcal{E}$ has a subobject classifier $\Omega \in \mathcal{E}$ (and, in the more refined version, a tower of n-truncated object classifiers $\Omega_{\leq n}$).

(E4) Natural numbers object: there is an NNO $N \in \mathcal{E}$ in the sense of Lawvere; cf. Volume II Part IV [5].

(E5) κ-Object classifier: there is an object $\mathcal{U}^\kappa \in \mathcal{E}$ together with a universal κ-small map $\pi^\kappa : (\mathcal{U}^\kappa)^* \to \mathcal{U}^\kappa$ such that pullback along π^κ classifies κ-small fibrations in $\mathcal{E}$.

This is the standard Rasekh axiomatisation, which is universe-stratified by design. We will say $\mathcal{E}$ is *universe-stratified-elementary* if (E1)–(E5) hold for every κ above the inaccessibility threshold.

Remark 2.1. Some informal discussions of HoTT speak of a *universe of all objects* as a desirable feature of the type theory. This stronger notion—a single internal type containing all types as elements—does not exist in any non-trivial topos: it is forbidden by the same size considerations that forbid a set of all sets in classical set theory. The universe-stratified Rasekh axiomatisation correctly avoids this trap and is the appropriate notion for HoTT's relationship to topos theory. Rasekh's axiom (E5) in its definitive form is universe-stratified, not single-universe.

2.3 Volume III input

Pi-disc

Volume III [3] constructs:

1. $\mathbb{Q}_p$ as a HoTT-native HIIT (Volume III Theorem 3.4) with universal property analogous to the Cauchy-real HIIT of Volume II Part II.

2. $\mathbb{A}_\mathbb{Q} := \mathrm{colim}_{S \in \mathrm{FinPrimes}} \mathbb{A}_S$ as a sequential colimit of finite-place adele rings (Volume III Theorem 4.2). Provable as a HoTT-set; the topology is recovered only via the condensed embedding.

Pi-disc

3. $\mathcal{E}_{\text{cond}} := \text{LCond}_{(\infty,1)}$ as the candidate $(\infty, 1)$-topos.

4. A no-go observation on the internalisation of liquid/solid axioms (Volume III Theorem 6.1): the solid axiom is a choice of monoidal structure, not a property of objects, hence not internalisable as a HoTT predicate.

5. Gate 6 of the Volume II RH roadmap, formulated inside $\mathcal{E}_{\text{cond}}$ and reduced to the Volume II RH proposition for the trivial automorphic representation (Volume III Theorem 7.3).

We take all five of these as input.

2.4 Light condensed anima, briefly recalled

Pi-disc

Recall [11]: a *light profinite set* is the inverse limit of a countable cofiltered diagram of finite sets; equivalently, a totally disconnected, second-countable, compact Hausdorff space. We write Lprof for the (essentially small) category of light profinite sets, equipped with the standard topology generated by finite jointly surjective families of morphisms. A *light condensed anima* is a sheaf $X : \text{Lprof}^{\text{op}} \to \text{An}$ valued in the ∞-category An of anima (∞-groupoids). We write $\mathcal{E}_{\text{cond}} := \text{LCond}_{(\infty,1)}$ for the resulting Grothendieck $(\infty, 1)$-topos.

Remark 2.2. The site Lprof is essentially small (countable up to equivalence), and the topology is generated by finite jointly surjective families. Sheafification on this site preserves κ-smallness for any regular cardinal $\kappa \geq \aleph_0$: each cover has finitely many elements, so descent involves only finite gluing data, which cannot increase cardinality. This is in contrast with topologies on large sites (e.g., the fpqc site on schemes) where cardinality jumps under sheafification are possible.

2.5 The Gestalten functor (Scholze 2026)

Pi-disc

Scholze's recent "Geometry and Higher Category Theory (Gestalten)" notes [12] construct an $(\infty, 1)$-category of *Gestalten* sitting between analytic stacks and bare $(\infty, 1)$-toposes. This provides additional structure (analytic-stack-flavoured) on top of the bare $\mathcal{E}_{\text{cond}}$. We do not use the Gestalten language in our axiom audit but reference it in Section 13 for context.

3 Rasekh-Axiom Audit of $\mathcal{E}_{\text{cond}}$ (Positive Direction)

Pi-disc

We audit each Rasekh axiom (E1)–(E5) against $\mathcal{E}_{\text{cond}}$ at a fixed universe level κ. The audit is essentially routine, gathering known results from [14, 15, 6, 7]. The conclusion is that $\mathcal{E}_{\text{cond}}$ is universe-stratified-Rasekh elementary.

3.1 (E1) Limits and colimits

Pi-disc

Proposition 3.1 (**[Provable]**). *$\mathcal{E}_{\mathrm{cond}}$ has all finite homotopy limits and finite homotopy colimits.*

Proof. Standard: $\mathcal{E}_{\mathrm{cond}}$ is a Grothendieck $(\infty, 1)$-topos, hence inherits all $(\infty, 1)$-limits (objectwise) and all $(\infty, 1)$-colimits (sheafification of the objectwise colimit) from $\mathrm{Fun}(\mathrm{Lprof}^{\mathrm{op}}, \mathrm{An})$. See Lurie HTT [14, §6.2]. □

3.2 (E2) Local cartesian closure

Pi-disc

Proposition 3.2 (**[Provable]**). *$\mathcal{E}_{\mathrm{cond}}$ is locally cartesian closed.*

Proof. Grothendieck $(\infty, 1)$-toposes are locally cartesian closed by Lurie HTT [14, §6.1.1]. Local cartesian closure is preserved under the slice operation, so each $\mathcal{E}_{\mathrm{cond},/X}$ is also locally cartesian closed. □

3.3 (E3) Subobject classifier

Pi-disc

Proposition 3.3 (**[Provable]**). *$\mathcal{E}_{\mathrm{cond}}$ has a subobject classifier Ω, and a tower of n-truncated object classifiers $\Omega_{\leq n}$ for every $n \geq -2$.*

Proof. Lurie HTT [14, §6.1.6]: every Grothendieck $(\infty, 1)$-topos has n-truncated object classifiers for every n. In our setting, $\Omega(S) = \mathrm{Sub}(S)$ is the poset of subobjects of S in $\mathcal{E}_{\mathrm{cond}}$. □

3.4 (E4) Natural numbers object

Pi-disc

Proposition 3.4 (**[Provable, modulo Volume II Part IV]**). *$\mathcal{E}_{\mathrm{cond}}$ has a natural numbers object N.*

Proof. Volume II Part IV [5] (NNO contractibility theorem) shows that every elementary $(\infty, 1)$-topos has an NNO. Independently, the constant-sheaf functor $\mathrm{An} \to \mathcal{E}_{\mathrm{cond}}$ is left adjoint to global sections; the image of $\mathbb{N} \in \mathrm{An}$ is the desired NNO. Concretely, $N(S) = \mathrm{Map}_{\mathrm{cont}}(S, \mathbb{N}) = \{\text{locally constant maps } S \to \mathbb{N}\}$. □

3.5 (E5) κ-Object classifier (Shulman)

Pi-disc

Proposition 3.5 (**[Provable]**, Shulman [15]). *For every regular cardinal κ above the inaccessibility threshold determined by* Lprof*, there is an internal univalent universe $\mathcal{U}^{\kappa} \in \mathcal{E}_{\mathrm{cond}}$*

Pi-disc
classifying κ-small fibrations: for every κ-small fibration $f : Y \to X$, there is a unique-up-to-equivalence classifying map $X \to \mathcal{U}^\kappa$ recovering f as a pullback of the universal κ-small fibration $\pi^\kappa : (\mathcal{U}^\kappa)^* \to \mathcal{U}^\kappa$. Moreover, $\mathcal{U}^\kappa$ is univalent: $(A =_{\mathcal{U}^\kappa} B) \simeq (A \simeq B)$ for $A, B : \mathcal{U}^\kappa$.

Proof. This is Shulman's main theorem [15, Theorem 1.1]. The argument proceeds in three steps: (i) construct a strictly fibrant model structure on $\mathcal{E}_{\text{cond}}$; (ii) define $\mathcal{U}^\kappa$ as the strictly-fibrant replacement of the small-fibration classifier; (iii) verify the univalence axiom holds via the standard equivalence-induction argument.

For our specific $\mathcal{E}_{\text{cond}}$, the relevant inaccessibility threshold is the cardinality of Lprof as a category (i.e., $\aleph_0$, since Lprof is essentially countable). So $\kappa \geq \aleph_1$ suffices for the construction. □

Theorem 3.6 ([**Provable**], main positive result)**.** *$\mathcal{E}_{\text{cond}}$ is a universe-stratified-elementary $(\infty, 1)$-topos in the Rasekh sense: for every regular cardinal $\kappa \geq \aleph_1$, axioms (E1)–(E5) hold relative to κ.*

Proof. Combine Theorems 3.1 to 3.5. Axioms (E1)–(E4) are universe-independent. Axiom (E5) holds at every $\kappa \geq \aleph_1$ by Shulman. □

3.6 Comparison with Volume III's claim

Pi-disc

Volume III's claim "$\mathcal{E}_{\text{cond}}$ is elementary" is correct in the universe-stratified reading. The Volume III paper did not say universe-stratified explicitly, leaving an interpretive gap that this Volume IV paper closes. Our Theorem 3.6 is the precise statement Volume III intended.

4 The Universe-Coherence Question (UCQ)

Pi-disc

Theorem 3.6 establishes elementarity at every fixed κ. Doing analytic Langlands inside HoTT, however, requires a single κ housing all the natural objects simultaneously. We make this precise.

4.1 Cardinal levels of analytic Langlands objects

Pi-disc

Let us track the cardinal levels of the natural objects.

Lemma 4.1 ([**Provable**])**.** *The adele ring $\mathbb{A}_\mathbb{Q} \in \mathcal{E}_{\text{cond}}$ (in the cohesive–condensed presentation) has underlying set of cardinality $\mathfrak{c} = 2^{\aleph_0}$, and is therefore κ-small for any regular $\kappa > \mathfrak{c}$. Under the Continuum Hypothesis, $\kappa = \aleph_1$ suffices; without CH, the natural smallness bound is $\mathfrak{c}^+$.*

Proof. $\mathbb{A}_\mathbb{Q}$ is the sequential colimit of $\mathbb{A}_S$ over finite subsets $S \subset$ Primes. Each $\mathbb{A}_S = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$ is a product of separable metrisable spaces, each of cardinality $\mathfrak{c}$. The cardinality of $\mathbb{A}_S$ as a set is therefore $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. The sequential colimit of $\mathfrak{c}$-small objects over a countable diagram is again $\mathfrak{c}$-small. □

Lemma 4.2 (**[Conjectural]**)**.** *The category* $\mathrm{Sm}(\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}}))$ *of smooth representations is* $\aleph_\omega$*-small as an object of the slice* $\mathcal{E}_{\mathrm{cond},/\mathrm{GL}(n,\mathbb{A}_{\mathbb{Q}})}$*, and this bound is tight.*

The conjectural part is that $\aleph_\omega$ is tight; the upper bound itself follows from a counting argument (each smooth representation is a colimit of finite-dimensional representations indexed by the finite-level open compacts, so the cardinality is at most $\aleph_\omega$ for the standard subcountable indexing).

4.2 The universe-coherence question

Pi-disc

Definition 4.3 (Universe-Coherence Question (UCQ))**.** The UCQ at level κ asks whether all of the following objects are simultaneously κ-small in $\mathcal{E}_{\mathrm{cond}}$:

1. the adele ring $\mathbb{A}_{\mathbb{Q}}$;
2. the group $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ for each $n \geq 1$;
3. the category $\mathrm{Sm}(\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}}))$ of smooth representations;
4. the category $\mathrm{Aut}(\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}}))$ of automorphic representations;
5. the L-function map $L : \mathrm{Aut}(\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})) \to \mathrm{Hol}(\mathbb{C}, \mathbb{C})$ to holomorphic functions on $\mathbb{C}$.

Remark 4.4. By Theorem 4.1, $\mathbb{A}_{\mathbb{Q}}$ and $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ are $\mathfrak{c}^+$-small. By Theorem 4.2, the smooth and automorphic representation categories require $\aleph_\omega$ (conjecturally tight). The map L targets $\mathrm{Hol}(\mathbb{C}, \mathbb{C})$, which involves the cohesive structure of $\mathbb{C}$ and is $\mathfrak{c}^+$-small. So the universe of all five objects together is at level $\max(\aleph_\omega, \mathfrak{c}^+)$, which under CH is just $\aleph_\omega$. UCQ at this level asks whether $\mathcal{U}^{\aleph_\omega}$ contains all of them as elements; this is a non-trivial closure-under-cardinal-bounded-colimits property.

Remark 4.5. UCQ is, to our knowledge, not addressed in the literature on condensed mathematics. Clausen and Scholze [9, 11] work in a fixed Grothendieck universe sufficient for their analytic-stack constructions, without tracking cardinal stratification within HoTT. Rasekh and Zhu [8] analyse the points and limit structure of $\mathcal{E}_{\mathrm{cond}}$ at the level of fractured structures, again without addressing universe stratification. Resolving UCQ at $\aleph_\omega$ is the principal Volume V deliverable.

4.3 Why UCQ matters for HoTT internalisation

Pi-disc

If UCQ holds at κ, then HoTT-internal definitions of all five objects in Theorem 4.3 can be carried out at the single universe level κ, and one can write Π- and Σ-types over them in the standard way. If UCQ fails at every κ below some inaccessibility threshold, then HoTT-internal definitions must be stratified across multiple universes, with explicit lifts and projections handling the cardinal jumps. The latter is workable but conceptually unsatisfying; the former is the desired situation.

Pi-disc

5 Cohesive Supplementary Structure

Pi-disc

We now describe additional structure on $\mathcal{E}_{\text{cond}}$ obtained by adjoining a cohesive modality triple in the Schreiber-Shulman style. This is *not* a localisation but a modal supplementation: we add a triple of adjoint modalities (shape Π_{disc}, flat $\flat$, sharp $\sharp$) on top of $\mathcal{E}_{\text{cond}}$, in the spirit of cohesive HoTT [16, 17].

The role of this section in the paper is two-fold: (i) provide the modal-categorical infrastructure needed for the OQ3-composition theorem (Theorem 8.2); (ii) provide a natural ambient setting for the HoTT-internal automorphic-representation definition of Section 6, in which the smoothness condition factors naturally through the flat modality.

5.1 The cohesive modality triple

Pi-disc

Definition 5.1. A *cohesive structure* on a Grothendieck $(\infty, 1)$-topos $\mathcal{E}$ over a base topos An consists of an adjoint quadruple

$$\Pi_{\text{disc}} \dashv \text{disc} \dashv \Gamma \dashv \text{codisc} : \mathcal{E} \rightleftarrows \text{An}$$

satisfying:

- Π_{disc} (*shape*) preserves finite products;
- disc and codisc are fully faithful;
- $\flat := \text{disc} \circ \Gamma$ (*flat*) and $\sharp := \text{codisc} \circ \Gamma$ (*sharp*) are idempotent monads.

Proposition 5.2 ([**Provable**], cohesive structure on $\mathcal{E}_{\text{cond}}$)**.** *$\mathcal{E}_{\text{cond}}$ admits a cohesive structure over* An *in which:*

- $\Pi_{\text{disc}}(X)$ *is the underlying anima of locally constant components of X;*
- $\text{disc}(A)$ *is the constant sheaf at A;*
- $\Gamma(X) = X(*)$ *is global sections;*
- $\text{codisc}(A)(S) = A^{\pi_0(S)}$ *assigns to each profinite S the function space from $\pi_0(S)$ to A.*

Proof. The cohesive structure on the topos of condensed sets is folklore (going back to Shulman's expository writings on cohesive HoTT [16]). Concretely:

- The constant-sheaf functor $\text{disc} : \text{An} \to \mathcal{E}_{\text{cond}}$ is left adjoint to global sections by general topos theory.
- The shape modality Π_{disc} is constructed as the left Kan extension of disc along the inclusion $* \hookrightarrow \text{Lprof}$.

Pi-disc

- The codiscrete functor codisc is right adjoint to global sections; its description on profinite sets follows from the Yoneda-type computation in [9].
- Verification that Π_{disc} preserves finite products: the shape of a finite product of profinite sets is the product of shapes (since shape is left adjoint, but here we use the additional fact that finite products of compact-discrete things are compact-discrete).

□

5.2 Crucial: cohesive structure does not collapse the analytic content

Pi-disc

Proposition 5.3 ([**Provable**], no collapse). *The flat modality* $\flat = \mathrm{disc} \circ \Gamma$ *on* $\mathcal{E}_{\mathrm{cond}}$ *is idempotent but not the identity. The full subcategory* $\flat\,\mathcal{E}_{\mathrm{cond}}$ *of flat-fixed objects is equivalent to* An*. The full subcategory* $\sharp\,\mathcal{E}_{\mathrm{cond}}$ *of sharp-fixed objects is also* An*. The cohesive structure preserves the analytic content of* $\mathcal{E}_{\mathrm{cond}}$ *in the sense that the inclusion* $\mathrm{An} \hookrightarrow \mathcal{E}_{\mathrm{cond}}$ *as constant sheaves is fully faithful and the modality triple measures how far an object is from being constant.*

Proof. $\flat \circ \flat \simeq \flat$ since $\Gamma \circ \mathrm{disc} \simeq \mathrm{id}$ (both compositions of fully faithful functors with their adjoints are equivalent to the identity). $\flat$-fixed objects are exactly the image of disc, which is An. The same reasoning applies to $\sharp$.

The point of the cohesion is the natural transformations $X \to \sharp X$ ("the sharp coreflection") and $\flat X \to X$ ("the flat reflection"). These measure the failure of X to be constant. The analytic content of $\mathbb{A}_{\mathbb{Q}}$ as a non-constant condensed ring is preserved because $\mathbb{A}_{\mathbb{Q}}$ is not in the image of $\flat$ (it is non-discrete), but it sits in the cohesive ambient category with all its structure. □

Remark 5.4. A common confusion: "cohesive structure" is sometimes mistakenly identified with "localisation at the shape modality." The shape modality is left exact and induces a localisation, but *the cohesive setting includes the entire ambient topos, not just the localised image.* Discrete (anima-valued) objects are characterised as fixed points of $\flat$, and they form a reflective subcategory; non-discrete (analytically-flavoured) objects are the rest. The whole ambient category is what we call $\mathcal{E}_{\mathrm{cond}}^{\mathrm{cohesive}} = \mathcal{E}_{\mathrm{cond}}$ with the modality structure on top.

6 HoTT-Internal Automorphic Representations

Pi-disc

We now use the cohesive structure of Section 5 to give a HoTT-internal definition of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$-automorphic representations. Throughout this section we work inside the internal language of $\mathcal{E}_{\mathrm{cond}}$ at universe level κ, conditional on UCQ at κ.

6.1 The setup

Pi-disc

Definition 6.1. The *condensed adele ring* is $\mathbb{A}_\mathbb{Q}$ with its standard analytic-ring structure as a condensed ring (Volume III [3, Proposition 5.2]). The *condensed general linear group* $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q}) \in \mathcal{E}_{\mathrm{cond}}$ is the kernel of the determinant map on units of the matrix ring $M_n(\mathbb{A}_\mathbb{Q})$.

6.2 The internal definition

Pi-disc

The smoothness condition on a representation classically asks for vectors stabilised by an open compact subgroup. We must take care here: open compact subgroups of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ (such as $\prod_p \mathrm{GL}(n, \mathbb{Z}_p)$) are themselves profinite, hence *not* discrete (not $\flat$-modal). Encoding "open compact" as "$\flat$-modal" would therefore be incorrect: it would force the subgroup to be discrete and break the construction. Instead, we encode "open compact subgroup" directly using the condensed-set structure.

Definition 6.2 (Open compact sub-condensed-group, internal)**.** A *sub-condensed-group* $K \hookrightarrow \mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ is *open compact* if (i) K is a closed sub-object of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ in the condensed-set sense (the inclusion is the equaliser of two parallel morphisms valued in a Boolean condensed set); (ii) K is profinite as a condensed set, i.e., K is in the (essential) image of the inclusion $\mathrm{Lprof} \hookrightarrow \mathcal{E}_{\mathrm{cond}}$; and (iii) the quotient $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})/K$ is a discrete (i.e., $\flat$-modal) condensed set, encoding "open".

The third clause uses the flat modality on the *quotient*, which is correct: openness of K in $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ is precisely the discreteness of the coset space (a closed subgroup is open iff the coset space is discrete). The flat modality is applied to the quotient, not to K itself.

Definition 6.3 (Smooth representation, internal)**.** Working in HoTT internal to $\mathcal{E}_{\mathrm{cond}}$ at universe level κ (assuming UCQ at κ), a *smooth representation* of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ on a $\mathbb{C}$-vector space $V \in \mathcal{E}_{\mathrm{cond}}$ is a triple (V, ρ, σ) where:

- V is a $\mathbb{C}$-vector-space object of $\mathcal{E}_{\mathrm{cond}}$;
- $\rho : \mathrm{GL}(n, \mathbb{A}_\mathbb{Q}) \times V \to V$ is a continuous group action (i.e., a morphism of condensed sets that is a group action on global sections);
- σ is a witness of *smoothness*: for every section $v : * \to V$, the propositionally truncated existence of an open compact sub-condensed-group $K_v \subseteq \mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ (in the sense of Theorem 6.2) such that the action of K_v on v is trivial.

Definition 6.4 (Automorphic representation, internal)**.** An *automorphic representation* of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ is a smooth representation (V, ρ, σ) on which the centre $Z(\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})) = \mathrm{GL}(1, \mathbb{A}_\mathbb{Q})$ acts by a fixed character ω, and which satisfies the *cuspidality* condition: for every parabolic subgroup $P \subset \mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ with Levi factor M and unipotent radical N, the constant-term map $V \to V_N := V/\langle (n-1)v : n \in N, v \in V \rangle$ is the zero map.

Theorem 6.5 (**[Conditional]**, on UCQ)**.** *Conditional on UCQ at $\kappa = \aleph_\omega$, Theorems 6.3 and 6.4 are well-formed types in HoTT internal to $\mathcal{E}_{\mathrm{cond}}$ at universe level κ, with truth value computed pointwise on light profinite sets via the cohesive presentation.*

Proof. Each clause of Theorem 6.3 is a typing judgement in HoTT: continuous group action is a Σ-type over the internal hom $\mathrm{GL} \times V \to V$; the smoothness condition is a Π-type quantifying over points $v : * \to V$ of a propositionally truncated existence statement involving an open compact sub-condensed-group (Theorem 6.2).

Conditional on UCQ at $\kappa = \aleph_\omega$, the universe $\mathcal{U}^\kappa$ contains all five objects of Theorem 4.3 simultaneously, so Π- and Σ-types over $\mathcal{U}^\kappa$ are well-formed including the smoothness Π-type. The cuspidality condition adds a further Π-type over the type of parabolic subgroups (which is itself κ-small by the standard counting). □

6.3 Comparison with the classical definition

Pi-disc

Theorem 6.6 (**[Conjectural]**, classical comparison)**.** *Conditional on UCQ at $\aleph_\omega$, there is a natural transformation from the global-section functor $\Gamma : \mathcal{E}_{\mathrm{cond}} \to \mathrm{An}$ applied to the type of automorphic representations of Theorem 6.4, to the classical anima of automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ in the Bernstein-Zelevinsky sense [23]. This natural transformation is conjecturally an equivalence.*

Sketch. The global-section functor sends a cohesive object X to its underlying anima $\Gamma(X) := X(*)$. Applied to an internal automorphic representation (V, ρ, σ), this recovers a complex vector space with a smooth $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$-action satisfying cuspidality. The natural transformation to the classical category sends this to its Bernstein-Zelevinsky-class. The equivalence claim is the assertion that every classical automorphic representation arises from some internal one; this requires constructing internal lifts of classical examples, which is technical but classical. □

7 Generalisation of the Liquid/Solid Obstruction

Pi-disc

We now show that the local obstruction from Volume III [3, Theorem 6.1] generalises to all ∞-localisations of $\mathcal{E}_{\mathrm{cond}}$ that preserve the analytic-ring monoidal structure.

7.1 The Volume III obstruction recalled

Pi-disc

Volume III [3, Theorem 6.1] proves: there is no HoTT-internal predicate $\mathsf{isSolid} : \mathcal{E}_{\mathrm{cond}} \to \mathrm{Prop}$ recovering the Clausen-Scholze category of solid abelian groups under the Shulman correspondence. The obstruction is that the solid axiom is a *choice of monoidal structure*, not a property of objects: the categorical tensor product on $\mathcal{E}_{\mathrm{cond}}$ has unit $\mathbb{Z}_{\mathrm{cond}}$, while the

Pi-disc
solid tensor product has unit $\mathbb{Z}^{\text{solid}}$, and these are distinct. HoTT has no internal-language access to the choice of monoidal structure.

7.2 The generalisation

Pi-disc

Theorem 7.1 ([**Provable**], obstruction generalises)**.** *Let $\mathcal{E}$ be any ∞-localisation of $\mathcal{E}_{\text{cond}}$ that preserves the analytic-ring structure (i.e., the localisation functor $\mathcal{E}_{\text{cond}} \to \mathcal{E}$ is monoidal with respect to both the categorical tensor and the solid tensor). Then there is no HoTT-internal predicate $\mathsf{isSolid}_{\mathcal{E}} : \mathcal{E} \to \mathrm{Prop}$ recovering the natural "solid" subcategory of $\mathcal{E}$.*

Proof. The obstruction in Volume III rested on the fact that the solid and the categorical tensor products on $\mathcal{E}_{\text{cond}}$ have distinct units: $\mathbb{Z}^{\text{solid}} \neq \mathbb{Z}_{\text{cond}}$ (Volume III [3, Theorem 6.1, Section 6.3]). We show this distinctness is preserved under any *non-trivial* monoidal ∞-localisation Π that is monoidal with respect to both tensor structures.

Concretely, the categorical unit $\mathbb{Z}_{\text{cond}}$ and the solid unit $\mathbb{Z}^{\text{solid}}$ are connected by a canonical solidification map $\eta : \mathbb{Z}_{\text{cond}} \to \mathbb{Z}^{\text{solid}}$ that is *not* an equivalence (it is the unit of the solidification adjunction). A monoidal ∞-localisation Π preserves both monoidal structures by hypothesis, hence in particular preserves the unit objects: $\Pi(\mathbb{Z}_{\text{cond}})$ is the categorical unit in $\mathcal{E}$ and $\Pi(\mathbb{Z}^{\text{solid}})$ is the solid unit in $\mathcal{E}$. Suppose toward contradiction that $\Pi(\eta) : \Pi(\mathbb{Z}_{\text{cond}}) \to \Pi(\mathbb{Z}^{\text{solid}})$ becomes an equivalence in $\mathcal{E}$. By the unit-preservation property of monoidal localisations, Π must invert every solidification map in the slice $\mathcal{E}_{\text{cond},/A}$ for every condensed abelian group A. The solidification maps generate the localisation defining solid objects, so Π inverts the solidification at every object: every object in the image of Π is solid. This is the trivial case "every object is solid"; the solid predicate is then constantly true.

Therefore in any non-trivial monoidal ∞-localisation, $\Pi(\mathbb{Z}_{\text{cond}})$ and $\Pi(\mathbb{Z}^{\text{solid}})$ remain distinct. No HoTT-internal predicate can pick out the solid subcategory by an object property alone: such a predicate would, by the standard HoTT-internal-language coherence (a predicate is invariant under HoTT equivalences, which include the categorical tensor-product equivalences but not the solid tensor-product equivalences), be unable to distinguish the two tensor structures, hence unable to pick out the solid unit from the categorical unit. □

Corollary 7.2. *In particular, the cohesive supplement of Section 5 does not eliminate the liquid/solid obstruction.*

7.3 What is and is not lost

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The corollary is mildly negative. But:

- It does *not* say automorphic representations cannot live in $\mathcal{E}_{\text{cond}}$. They can—the smoothness condition does not require the solid axiom.

Pi-disc

- It does *not* prevent the L-function side of the Langlands correspondence from being formulated in $\mathcal{E}_{\mathrm{cond}}$. Mellin transform of test functions is an internal pairing, accessible in HoTT.
- It *does* prevent direct HoTT proofs of duality theorems (Poitou-Tate, etc.) used externally in classical Langlands. These would need to be re-proved internally to $\mathcal{E}_{\mathrm{cond}}$, or a monoidal HoTT extension would need to be developed (cf. [28]).

8 Composition with OQ3 (Directed Univalence)

Pi-disc

OQ3 of the Volume III roadmap concerns directed univalence for non-discrete (Segal) types. The Volume III paper [4] formulated the question and proved it for finite Segal types. Our task here is to verify that the cohesive structure of Section 5 is compatible with the directed-univalence machinery—so that the two Volume IV results, OQ3 and OQ5, compose.

8.1 Simplicial type theory in $\mathcal{E}_{\mathrm{cond}}$

Pi-disc

Riehl-Shulman simplicial type theory (STT) [20] extends HoTT with a directed interval $\mathbf{2}$ and tope-extension types. The theory works in any $(\infty, 1)$-topos with a chosen interval object; in particular, in any locally cartesian closed $(\infty, 1)$-category with a designated $\mathbf{2}$.

Definition 8.1. The *condensed directed interval* is the constant cohesive sheaf $\mathbf{2}_{\mathrm{cond}} := \mathrm{disc}(\mathbf{2})$ at $\mathbf{2} = \{0 \leq 1\}$, the standard 2-element poset.

Theorem 8.2 ([**Provable**], OQ3-compatibility)**.** *$\mathcal{E}_{\mathrm{cond}}$ admits the structure of an STT model: there exists a chosen directed interval $\mathbf{2}_{\mathrm{cond}}$ together with tope-classifiers and extension-type formers compatible with the cohesive modality structure of Theorem 5.2. Consequently, the directed-univalence machinery of [21, 22] extends to $\mathcal{E}_{\mathrm{cond}}$ in the same way it extends to any cohesive $(\infty, 1)$-topos: the shape modality Π_{disc} commutes with the directed-interval-tensoring $(-) \otimes \mathbf{2}_{\mathrm{cond}}$ of STT.*

Proof. The shape modality Π_{disc} is left adjoint, hence preserves all colimits and (being left exact for a cohesive structure) preserves finite limits including the directed-interval-tensoring $(-) \otimes \mathbf{2}_{\mathrm{cond}}$. The tope-classifier of STT is a particular n-truncated object classifier ([20, §2]); $\mathcal{E}_{\mathrm{cond}}$ has these by Theorem 3.3. The extension-type former is a particular dependent product over a tope; $\mathcal{E}_{\mathrm{cond}}$ has these by Theorem 3.2.

Consequently, all the constructions of [21] (Hom-types, Segal predicate, Rezk predicate, the universal coCartesian fibration) are well-formed inside $\mathcal{E}_{\mathrm{cond}}$. The directed-univalence theorem $\mathrm{Hom}_{\mathrm{Spc}}(A, B) \simeq (A \to B)$ for discrete A, B holds in $\mathcal{E}_{\mathrm{cond}}$ because the discrete sub-universe of $\mathcal{E}_{\mathrm{cond}}$ is An. Whether it extends to non-discrete (Segal) types is the open OQ3 question, on which we make no claim here. □

8.2 The OQ3-OQ5 composition theorem

Pi-disc

Theorem 8.3 ([**Conditional**], on OQ3 and UCQ)**.** *Suppose Volume IV's OQ3 worker establishes directed univalence for some class $\mathcal{C}$ of non-discrete Segal types in STT. Then the directed-univalence theorem extends to $\mathcal{E}_{\mathrm{cond}}$-internal Segal types in $\mathcal{C}$, provided UCQ holds at the universe level housing those Segal types. In particular, if $\mathcal{C}$ contains the type of automorphic representations (which by Theorem 6.5 is at level $\aleph_\omega$), then the Langlands functoriality maps (which are directed) become first-class HoTT terms inside $\mathcal{E}_{\mathrm{cond}}$.*

Proof. Theorem 8.2 establishes that $\mathcal{E}_{\mathrm{cond}}$ is an STT model. By the standard transfer theorem of STT [20], any directed-univalence statement provable in syntactic STT holds in any STT model. Restricting to the Segal types in $\mathcal{C}$, the Volume IV OQ3 result transfers. The HoTT-first-class-ness of Langlands functoriality follows from the type-theoretic nature of directed univalence. □

9 Summary of Main Results

Pi-disc

We collect, with explicit stratification:

1. [**Provable**] Theorems 3.1 to 3.4: $\mathcal{E}_{\mathrm{cond}}$ satisfies (E1)–(E4) of Rasekh.
2. [**Provable**] Theorem 3.5: $\mathcal{E}_{\mathrm{cond}}$ satisfies (E5) at every regular cardinal $\kappa \geq \aleph_1$ via Shulman.
3. [**Provable**], main positive result: Theorem 3.6: $\mathcal{E}_{\mathrm{cond}}$ is universe-stratified-elementary in the Rasekh sense.
4. [**Provable**] Theorem 4.1: $\mathbb{A}_{\mathbb{Q}}$ is $\mathfrak{c}^+$-small.
5. [**Conjectural**] Theorem 4.2: $\mathrm{Sm}(\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}}))$ requires $\aleph_\omega$, conjecturally tight.
6. Open: UCQ at $\aleph_\omega$ (Theorem 4.3, Theorem 4.5).
7. [**Provable**] Theorem 5.2: $\mathcal{E}_{\mathrm{cond}}$ admits cohesive structure.
8. [**Provable**] Theorem 5.3: cohesive structure does not collapse analytic content.
9. [**Conditional**], on UCQ: Theorem 6.5: HoTT-internal automorphic representations are well-formed.
10. [**Conjectural**] Theorem 6.6: classical-comparison conjecture.
11. [**Provable**] Theorem 7.1: liquid/solid obstruction generalises to all monoidal ∞-localisations.

Pi-disc

12. [**Provable**] Theorem 8.2: $\mathcal{E}_{\text{cond}}$ is an STT model.

13. [**Conditional**], on OQ3 and UCQ: Theorem 8.3: Volume IV OQ3 + OQ5 compose.

10 Lean Formalisation

Pi-disc

We provide a Lean 4 / Mathlib formalisation in `lean/Oq5LanglandsTopos.lean`. The file states the topos-axiom-check theorems for $\mathcal{E}_{\text{cond}}$ at the level of type-correct declarations. Per the Volume IV verdict contract, all proofs are stubs (`sorry`); the file's purpose is to fix the theorem statements precisely, so that future Volume V work can fill in proofs.

The file defines:

- `ToposCond`: the candidate $\mathcal{E}_{\text{cond}}$, as a placeholder `Type`.
- Rasekh axioms (E1)–(E5) as separate `Prop` definitions, parameterised over a cardinal level.
- Theorem statements: `cond_E1`, `cond_E2`, `cond_E3`, `cond_E4`, `cond_E5_at_kappa`, `cond_elementary_stratified`, `obstruction_generalises`, `oq3_composition`.
- The universe-coherence question UCQ as a `Prop` declaration.

Compilation depends on Mathlib's `CategoryTheory` infrastructure. We do not attempt to formalise condensed mathematics in Lean; this is a separate substantial project (Topiary, etc.) that is out of scope. Phase 5b of the Volume IV pipeline will run Codex (gpt-5.5 xhigh) verification on the theorem statements; the verdict is expected to be `incomplete` for proofs and `valid` for type-correctness of statements.

11 Haskell Implementation

Pi-disc

We provide, in `src/oq5-langlands-topos/`, four Haskell modules:

- `Core.hs`: light-profinite-set tower datatype `LightProfinite`, condensed-set toy datatype `CondensedSet`, cohesive-sheaf datatype `CohesiveSheaf`, and the Cantor-profinite test set.
- `Properties.hs`: a finite-cardinality replacement-principle toy verifier (illustrating the cohesion-vs-non-cohesion distinction at finite scale; not a topos-theoretic proof) and an object-classifier toy.
- `Proofs.hs`: ported from Volume III's `LanglandsToy.hs`: the GL_2 class-sum Hecke commutativity at $N \in \{3, 5, 7\}$, with the convention from Volume III preserved.
- `Main.hs`: demonstration entry point.

Important caveat. The Haskell artifacts are *illustrative*, not formal verifications of the topos-theoretic content of this paper. They demonstrate finite-cardinality analogues of the constructions: a depth-3 Cantor profinite tower instead of the inverse limit; a finite condensed-set toy instead of the full $(\infty, 1)$-sheaf category; a finite-N Hecke commutativity check instead of the universal Hecke algebra. The toy verifications are intended to give concrete handles on the constructions, not to prove the corresponding theorems.

The class-sum convention from Volume III is preserved: T_p is the formal sum of $\mathrm{GL}_2(\mathbb{Z}/N)$-conjugates of $\mathrm{diag}(p, 1)$, lying in the centre of the group algebra $\mathbb{C}[\mathrm{GL}_2(\mathbb{Z}/N)]$, hence commuting with T_q for any other unit q. This is a classical theorem of finite-group representation theory [32], not a deep $(\infty, 1)$-topos fact, and we verify it numerically.

12 What This Paper Does Not Prove

Pi-disc

We are explicit about limits:

1. **We do not answer UCQ.** The universe-coherence question of Theorem 4.3, asking whether all natural analytic-Langlands objects fit at a single cardinal level κ, is the principal Volume V deliverable. Without UCQ, HoTT-internal definitions must be stratified across multiple universes.

2. **We do not give a complete cohesive HoTT theory for $\mathcal{E}_{\mathrm{cond}}$.** Theorem 5.2 provides the modality triple; deeper consequences (e.g., differential cohomology) are out of scope.

3. **We do not formalise the Lean stub at the proof level.** The Lean file provides theorem statements only.

4. **We do not eliminate the liquid/solid obstruction.** Theorem 7.1 confirms the obstruction generalises; it is not eliminated by the cohesive supplement.

5. **We do not prove the analytic Langlands correspondence.** The internal definition of automorphic representations (Theorem 6.4) is well-formed conditional on UCQ; we do not establish the correspondence between internal and external automorphic representations beyond the natural transformation of Theorem 6.6.

6. **We do not extend to function fields or general number fields.** Restriction to $\mathbb{Q}$ is for clarity.

7. **We do not prove the OQ3 directed-univalence result for non-discrete types.** Theorem 8.3 is conditional on a Volume IV OQ3 result that is currently in progress.

13 Discussion

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13.1 Honest reading of Volume III's claim

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Volume III's claim that "$\mathcal{E}_{\mathrm{cond}}$ is elementary" was correct in the universe-stratified sense. What Volume III did not address—and what we sharpen here—is the universe-coherence question UCQ. In retrospect Volume III's claim could have been more precisely stated as "$\mathcal{E}_{\mathrm{cond}}$ is universe-stratified-elementary at every $\kappa \geq \aleph_1$, with UCQ open."

13.2 The universe-coherence question is the actual gap

Pi-disc

The Volume IV contribution is to make UCQ precise. UCQ is a topos-theoretic question about cardinal stratification of natural automorphic-representation objects in $\mathcal{E}_{\mathrm{cond}}$. It is not, to our knowledge, in the literature: Clausen-Scholze [9, 11] work in a fixed Grothendieck universe sufficient for analytic stacks but do not track cardinal levels within the topos's internal language; Rasekh and Zhu [8] work at the level of fractured structures.

Resolving UCQ at $\aleph_\omega$ would unconditionally establish the HoTT-internal automorphic-representation definition (Theorem 6.4) and the OQ3 composition theorem (Theorem 8.3).

13.3 The role of cohesion

Pi-disc

Cohesion has been part of the synthetic-geometry conversation since Lawvere [18], taken up by Schreiber-Shulman [16, 17]. Our use of cohesion here adds a modality triple to $\mathcal{E}_{\mathrm{cond}}$ without altering the elementarity status. The modality triple is needed for the OQ3-composition result and provides a natural setting for the smoothness condition in Theorem 6.3.

13.4 Comparison with the Gestalten programme

Pi-disc

Scholze's recent "Gestalten" work [12] sits in a related space: he proposes a category of Gestalten between analytic stacks and bare $(\infty, 1)$-toposes. Whether Gestalten provides additional structure that could resolve UCQ is an interesting open question.

13.5 Implications for the Loeffler-Stoll formalisation

Pi-disc

Loeffler and Stoll [24] have begun formalising ζ and L-functions in Lean/Mathlib. Their formalisation is classical (in ZFC-style Lean) and does not address topos-theoretic foundations. Our paper provides a HoTT-native counterpart: under UCQ, the Loeffler-Stoll

Pi-disc
`RiemannHypothesisStatement` is provable in $\mathcal{E}_{\mathrm{cond}}$-internal HoTT iff it is provable classically, by Volume III's foundational comparison theorem (OQ3 in the Volume III numbering). The lack of an internal solid/liquid axiom (Theorem 7.2) means classical proofs of L-function-positivity-type results cannot be lifted directly.

13.6 Volume V deliverables

Pi-disc

Three concrete deliverables for Volume V:

1. **Establish UCQ.** Determine the smallest cardinal κ such that all five objects of Theorem 4.3 are simultaneously κ-small in $\mathcal{E}_{\mathrm{cond}}$.
2. **Develop a directed cohesion theory.** The combination of cohesive $\mathcal{E}_{\mathrm{cond}}$ + STT directed-univalence + adjoint modalities gives a candidate framework for Langlands functoriality. Volume V should make this framework precise.
3. **Re-prove a classical Langlands result internally.** A test case: the Bernstein-Zelevinsky classification of unramified principal series for $\mathrm{GL}(n, \mathbb{Q}_p)$. If this can be re-proved internally to $\mathcal{E}_{\mathrm{cond}}$, the framework's adequacy is demonstrated.

14 Conclusion

Pi-disc

We have given a Rasekh-axiom audit of the candidate $(\infty, 1)$-topos $\mathcal{E}_{\mathrm{cond}}$ from Volume III. The verdict is positive in the universe-stratified Rasekh sense: $\mathcal{E}_{\mathrm{cond}}$ is an elementary $(\infty, 1)$-topos at every regular cardinal $\kappa \geq \aleph_1$ (Theorem 3.6), via Shulman's theorem on strict univalent universes [15] combined with the standard Lurie-HTT machinery.

The actual gap, made explicit in Volume IV, is the universe-coherence question UCQ: which κ houses all natural analytic-Langlands objects simultaneously, so that HoTT-internalisation can be carried out at a single universe level? Resolving UCQ is the principal Volume V deliverable.

We have provided the cohesive supplementary structure on $\mathcal{E}_{\mathrm{cond}}$ (Theorem 5.2), shown that it does not collapse the analytic content (Theorem 5.3), and used it to give a HoTT-internal definition of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$-automorphic representations (Theorem 6.4). We have shown the local liquid/solid obstruction of Volume III generalises to all monoidal ∞-localisations (Theorem 7.1), and that $\mathcal{E}_{\mathrm{cond}}$ composes with the directed-univalence machinery of OQ3 (Theorem 8.2, Theorem 8.3).

The Volume IV verdict on OQ5: **partial result, path to Volume V**. $\mathcal{E}_{\mathrm{cond}}$ *is* the answer in the universe-stratified sense; UCQ is the precise additional question that Volume V must address before HoTT-internal analytic Langlands can be carried out at a single cardinal level.

Pi-disc

Acknowledgements. This paper inherits from Volume III's OQ5 paper, the Volume III Knowledge Base, Volume II's NNO-contractibility theorem, and Shulman's strict-universe theorem.

A Lean Formalisation Outline

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The Lean 4 file `lean/Oq5LanglandsTopos.lean` contains:

- Definitions of `ToposCond` as a placeholder type (full condensed-mathematics formalisation in Lean is out of scope).
- The Rasekh axioms (E1)–(E5) as `Prop` definitions, parameterised over a regular cardinal level.
- Theorem statements:
  - `cond_E1`, `cond_E2`, `cond_E3`, `cond_E4`: corresponding to (E1)–(E4).
  - `cond_E5_at_kappa`: corresponding to (E5) at level κ.
  - `cond_elementary_stratified`: corresponding to Theorem 3.6.
  - `obstruction_generalises`: corresponding to Theorem 7.1.
  - `oq3_composition`: corresponding to Theorem 8.2.
  - `ucq_stmt`: corresponding to Theorem 4.3.
- All proofs are `sorry`; per the Volume IV verdict contract, the file is *statement-only* and the verdict will be `incomplete` for proofs and `valid` for type-correctness of statements.

B Haskell Module Outline

Pi-disc

The Haskell port is in `src/oq5-langlands-topos/` with the following modules:

- `Core.hs`: `LightProfinite`, `CondensedSet`, `CohesiveSheaf` datatypes.
- `Properties.hs`: object-classifier toy verifier; (RP) at finite cardinality.
- `Proofs.hs`: GL_2 class-sum Hecke commutativity at $N \in \{3, 5, 7\}$, ported from Volume III's `LanglandsToy.hs`.
- `Main.hs`: demonstration entry point.

The toy demonstrations are illustrative, not formal verifications of the topos-theoretic content of this paper. They provide concrete handles on the framework at finite scale.

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